



Inviscid Large deviation principle and the 2D Navier Stokes equations with a free boundary condition

Hakima Bessaih, Annie Millet

► To cite this version:

Hakima Bessaih, Annie Millet. Inviscid Large deviation principle and the 2D Navier Stokes equations with a free boundary condition. SIAM Journal on Mathematical Analysis, 2012, 44 (3), pp.1861-1893. 10.1137/110827235 . hal-00537662v2

HAL Id: hal-00537662

<https://hal.science/hal-00537662v2>

Submitted on 26 Apr 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LARGE DEVIATIONS AND THE ZERO VISCOSITY LIMIT FOR 2D STOCHASTIC NAVIER STOKES EQUATIONS WITH FREE BOUNDARY

HAKIMA BESSAIH AND ANNIE MILLET

ABSTRACT. Using a weak convergence approach, we prove a LPD for the solution of 2D stochastic Navier Stokes equations when the viscosity converges to 0 and the noise intensity is multiplied by the square root of the viscosity. Unlike previous results on LDP for hydrodynamical models, the weak convergence is proven by tightness properties of the distribution of the solution in appropriate functional spaces.

Keywords: Models of turbulence, viscosity coefficient and Navier-Stokes equations, Euler equation, stochastic PDEs, Radonifying operators, large deviations.

Mathematics Subject Classification 2000: Primary 60H15, 60F10; 60H30; Secondary 76D06, 76M35.

1. INTRODUCTION

The vanishing viscosity limit for solutions of Navier-Stokes equations is a singular limit, that means that the type of the equation may change in the limit. Singular limits are ubiquitous in applied mathematics and correspond to physical reality. In bounded domains, the vanishing viscosity limit shows a physical phenomena called boundary layers. The Navier-Stokes equations are second-order differential equations and require Dirichlet boundary conditions, while the Euler equation only requires for the particle paths to be tangent to the boundary. At present the problem of vanishing viscosity limit is open even in two dimensions in bounded domains, while more progress has been made in the study of the limit when there is no boundary or when we impose particular boundary conditions, like the one we are considering in this paper. There are two distinct concepts of vanishing viscosity limit. The finite-time, zero viscosity limit of solutions of the Navier-Stokes with a fixed initial datum and with time t in some finite interval $[0, T]$. By contrast, in the infinite-time zero-viscosity limit, long-time averages of functionals of the solutions are considered first at fixed ν . These are represented by measures μ^ν in functions spaces and the zero-viscosity limit $\lim_{\nu \rightarrow 0} \mu^\nu$ is then studied. The two kinds of limits are not the same.

In the present paper, we are dealing with flows described by stochastic Navier Stokes equations in dimension 2 of the following form

$$\frac{\partial u_\nu(t)}{\partial t} - \nu \Delta u_\nu(t) + (u_\nu(t) \cdot \nabla) u_\nu(t) = -\nabla p + G_\nu(t, u_\nu(t)) \frac{\partial W(t, \cdot)}{\partial t}, \quad (1.1)$$

in an open bounded domain D of \mathbb{R}^2 with a smooth boundary ∂D which satisfies the locally Lipschitz condition see [1]. Here, u_ν is the velocity of the fluid, $\nu > 0$ is its viscosity, p denotes the pressure, W is a Gaussian random field white in time, subject to the restrictions imposed below on the space correlation and G_ν is an operator acting on the solution. The velocity field u_ν is subject to the incompressibility condition

$$\nabla \cdot u_\nu(t, x) = 0, \quad t \in [0, T], \quad x \in D, \quad (1.2)$$

and to the boundary condition for every $t \in [0, T]$

$$u_\nu(t, \cdot) \cdot n = 0 \quad \text{and} \quad \text{curl } u_\nu(t, \cdot) = 0 \quad \text{on } \partial D, \quad (1.3)$$

n being the unit outward normal to ∂D . The initial condition is the function ζ defined by:

$$u_\nu(0, x) = \zeta(x), \quad \forall x \in D. \quad (1.4)$$

We are interested in the asymptotic properties of the distribution of the process $u_\nu(t, \cdot)$ as the viscosity goes to 0. More precisely, the aim of the present paper is to prove a Large Deviation Principle (LDP) for the stochastic 2D Navier Stokes equations (1.1) when the viscosity coefficient $\nu \rightarrow 0$ and the noise W is multiplied by the square root of the viscosity, in order to be in the Freidlin-Wentzell setting. A similar idea has been pursued by S. B. Kuksin in [28], where he studied the convergence of the invariant measure of the equation (1.1) when it is driven by an additive degenerate noise. Indeed, Kuksin establishes asymptotic properties of this invariant measure when the viscosity is small. In this paper, we study the exponential concentration of the distribution of the process $u_\nu(t, \cdot)$ for a fixed t , when the viscosity decays to zero; we hope to be able to extend this study for stationary solutions.

Several recent papers have studied a LDP for the distribution of the solution to a hydro-dynamical stochastic evolution equation. We refer to [39] for the 2D Navier-Stokes equations, [23] for the Boussinesq model, [18] for more general hydro-dynamic models, [38] for tamed 3D Navier Stokes equations. All the above papers consider an equation with a (fixed) positive viscosity coefficient and study the exponential concentration to a deterministic model when the noise intensity is multiplied by a coefficient $\sqrt{\epsilon}$ which converges to 0. They deal with a multiplicative noise and use the weak convergence approach of LDP, based on the Laplace principle, developed by P. Dupuis and R. Ellis in [24].

Reference [6] dealt with a simpler equation driven by a multiplicative noise and a vanishing viscosity coefficient, that is a shell model of turbulence. Under certain conditions on the initial condition and the operator acting on the noise, this equation is well posed in $\mathcal{C}([0, T]; V)$ where V is a Hilbert space similar to $H^{1,2}$. A LDP was proved for a weaker topology, that of $L^2(0, T; \mathcal{H})$, where \mathcal{H} is a subspace of V similar to $H^{\frac{1}{2},2}$, with the same scaling between the "viscosity" and the square of the noise intensity. The technique used was again the weak convergence approach. To our knowledge, this was the first paper that proved a LDP when the coefficient in front of the noise term depends on the viscosity and converges to 0. Let us point out that the study of the inviscid limit is an important step towards understanding turbulent fluid flows in general. Let us also refer to the paper of M. Mariani [32], where a "nonviscous" scalar equation is considered in the context of conservation laws. However the techniques used in that paper are completely different from the ones used here and in [6].

In this paper, we will generalize our result to the Navier Stokes equations (1.1) in a bounded domain of \mathbb{R}^2 ; this is technically more involved. Here, the family $(G_\nu, \nu > 0)$ of operators is of the form $G_\nu = \sqrt{\nu}\sigma_\nu$, where the family σ_ν converges to σ_0 in an appropriate topology as $\nu \rightarrow 0$. Similarly, we can deal with a more general family $(\sigma_\nu, \nu > 0)$ of gradient type converging to some more regular operator σ_0 which is no longer of gradient type. Gradient type noise is an active topic of research for turbulent flows; see e.g. [35] and the references therein. However, in order to focus on the main ideas of the inviscid limit and avoid heavy technical computations, we choose to work with simpler σ_ν . Note that the rate function in this framework is described by the solution to a deterministic

"controlled" Euler equation

$$\frac{\partial u(t)}{\partial t} + (u(t) \cdot \nabla)u(t) = -\nabla p + \sigma_0(t, u(t))h(t), \quad (1.5)$$

with the same incompressibility and boundary conditions, where h denotes an element of the Reproducing Kernel Hilbert Space (RKHS) of the noise. This equation is a deterministic counterpart of the stochastic Euler equation studied by [4] in the case of additive noise, [11] and [5] when the noise is multiplicative. There is an extensive literature for the deterministic Euler equation in dimension 2. We refer to [2], [27], [40] and the references therein and [3] for a survey paper.

The technique we use is again the weak convergence approach and will require to prove well posedness and apriori bounds of the solution to (1.5) in the space $\mathcal{C}([0, T]; L^2) \cap L^\infty(0, T; H^{1,q})$ for all $q > 2$ and for a more regular initial condition. Thus, we are able to prove the LDP in a "non-optimal" space for the Navier Stokes equations with positive viscosity, namely $L^2(0, T; \mathcal{H})$, where \mathcal{H} is a Hilbert interpolation space between H and V similar to that in [6]. This is due to the fact that the Euler equation has no regularizing effect on the solutions and stronger conditions are required in order to have uniqueness of the solution; this forces us to work with non Hilbert Sobolev spaces $H^{1,q}$ for $q \in (2, \infty)$ and to require that the diffusion coefficient σ is both trace class and Radonifying. Indeed, some apriori estimates have to be obtained in general Sobolev spaces uniformly in the "small" viscosity $\nu > 0$ for the stochastic Navier Stokes equations (1.1) when the noise W is multiplied by $\sqrt{\nu}$ and shifted by a random element of its RKHS.

Let us finally point out that, even if the problem solved here is similar to that in [6], the final step is quite different. Indeed, unlike all the references on LDP for hydrodynamical models, the weak convergence is proven using a tightness argument and not by means of the convergence in L^2 of a properly localized sequence. Unlike in [6], no time increment has to be studied and no Hölder regularity of the map $\sigma(\cdot, u)$ has to be imposed. Let us also point out that we replace the classical homogenous Dirichlet boundary conditions by the free boundary one. Working with the classical homogeneous Dirichlet boundary condition would lead to some boundary layers problems that are beyond the scope of this paper. For more details and explanations about the free boundary condition (1.3) we refer to [41]. Let us also mention that all our results can be proved for the stochastic Navier-Stokes equations with periodic conditions.

The paper is organized as follows: In section 2 we describe the model and establish apriori estimates in the Hilbert spaces L^2 and $H^{1,2}$ similar to known ones, except for two things: the boundary conditions are slightly different, and we have to prove estimates uniform in a "small" viscosity ν . Section 3 deals with the inviscid problem in $C([0, T]; L^2) \cap L^\infty(0, T; H^{1,q})$. Section 4 proves apriori bounds of the NS equations in $H^{1,q}$ and section 5 establishes the large deviations results. Finally, some technical results on Radonifying and Nemytski's operators are gathered in the Appendix.

2. DESCRIPTION OF THE MODEL

For every $\nu > 0$, we consider the equations of Navier-Stokes type

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + G_\nu(t, u) \frac{\partial W}{\partial t}, & \text{in } [0, T] \times D, \\ \nabla \cdot u = 0, & \text{in } [0, T] \times D, \\ \text{curl } u = 0 \text{ and } u \cdot n = 0 & \text{on } [0, T] \times \partial D, \\ u|_{t=0} = \zeta, & \text{in } D, \end{cases} \quad (2.1)$$

where $\text{curl } u = D_1 u_2 - D_2 u_1$.

2.1. Notations and hypothesis. Let \mathcal{V} be the space of infinitely differentiable vector fields u on D with compact support strictly contained in D , satisfying $\nabla \cdot u = 0$ in D and $u \cdot n = 0$ on ∂D . Let us denote by H the closure of \mathcal{V} in $L^2(D; \mathbb{R}^2)$, that is

$$H = \left\{ u \in [L^2(D)]^2; \nabla \cdot u = 0 \text{ in } D, u \cdot n = 0 \text{ on } \partial D \right\}.$$

The space H is a separable Hilbert space with the inner product inherited from $[L^2(D)]^2$, denoted in the sequel by (\cdot, \cdot) and $\|\cdot\|_H$ denotes the corresponding norm. For every integer $k \geq 0$ and any $q \in [1, \infty)$, let $W^{k,q}$ denote the completion of the set of $\mathcal{C}_0^\infty(\bar{D}, \mathbb{R})$ or of $\mathcal{C}_0^\infty(\bar{D}, \mathbb{R}^2)$ with respect to the norm

$$\|u\|_{W^{k,q}} = \left(\sum_{|\alpha| \leq k} \int_D |\partial^\alpha u(x)|^q dx \right)^{\frac{1}{q}}.$$

To ease notations, let $\|\cdot\|_q := \|\cdot\|_{W^{0,q}}$. For $k < 0$ and $q^* = q/(q-1)$, let $W^{-k,q^*} = (W^{k,q})^*$.

Here, for a multi-index $\alpha = (\alpha_1, \alpha_2)$ we set $\partial^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$. For a non-negative real number $s = k + r$, where k is an integer and $0 < r < 1$, and for any $q \in [1, \infty)$, let $W^{s,q}$ denote the completion of the set of $\mathcal{C}_0^\infty(\bar{D}, \mathbb{R})$ or of $\mathcal{C}_0^\infty(\bar{D}, \mathbb{R}^2)$ with respect to the norm defined by:

$$\|u\|_{W^{s,q}}^q = \|u\|_{W^{k,q}}^q + \sum_{|\alpha|=k} \int_D \int_D \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^q}{|x - y|^{2+2r}} dx dy.$$

Given $0 < \alpha < 1$, let $W^{\alpha,p}(0, T; H)$ be the Sobolev space of all $u \in L^p(0, T; H)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+\alpha p}} dt ds < \infty.$$

Let us set $H^{k,q} = W^{k,q} \cap H$ for any $k \in [0, +\infty)$ and $q \in [2, \infty)$; the set $H^{k,q}$ is endowed with the norm inherited from that of $W^{k,q}$ and denoted by $\|\cdot\|_{H^{k,p}}$. Let $V = H^{1,2}$, that is the subspace of H defined as follows:

$$V = \left\{ u \in W^{1,2}(D; \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } D, u \cdot n = 0 \text{ on } \partial D \right\}.$$

The space V is a separable Hilbert space with the inner product $((\cdot, \cdot))$ inherited from that of $W^{1,2}(D; \mathbb{R}^2)$ and $\|\cdot\| := \|\cdot\|_V$ denotes the corresponding norm, defined for $u, v \in V$ by:

$$\|u\|^2 = ((u, u)), \text{ and } ((u, v)) = \int_D [u(x) \cdot v(x) + \nabla u(x) \cdot \nabla v(x)] dx.$$

Identifying H with its dual space H' , and H' with the corresponding natural subspace of the dual space V' , we have the Gelfand triple $V \subset H \subset V'$ with continuous dense injections. We denote the dual pairing between $u \in V$ and $v \in V'$ by $\langle u, v \rangle$. When $v \in H$, we have $(u, v) = \langle u, v \rangle$. Let $b(\cdot, \cdot, \cdot) : V \times V \times V \rightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u, v, z) = \int_D (u(x) \cdot \nabla v(x)) \cdot z(x) dx.$$

It is well known that there exists a continuous bilinear operator $B(\cdot, \cdot) : V \times V \rightarrow V'$ such that $\langle B(u, v), z \rangle = b(u, v, z)$, for all $z \in V$. By the incompressibility condition, for $u, v, z \in V$ we have (see e.g. [30] or [2])

$$\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle \quad \text{and} \quad \langle B(u, v), v \rangle = 0. \quad (2.2)$$

Furthermore, there exists a constant C such that for any $u \in V$,

$$\|B(u, u)\|_{V'} \leq C \|u\|_H \|u\|. \quad (2.3)$$

Let $a(\cdot, \cdot) : V \times V \longrightarrow \mathbb{R}$ be the bilinear continuous form defined in [2] as

$$a(u, v) = \int_D \nabla u \cdot \nabla v - \int_{\partial D} k(r)u(r) \cdot v(r)dr,$$

where $k(r)$ is the curvature of the boundary ∂D at the point r , and we have the following estimates (see [29] for details):

$$\int_{\partial D} k(r)u(r) \cdot v(r)dr \leq C\|u\|\|v\|, \quad (2.4)$$

and for any $\epsilon > 0$ there exists a positive constant $C(\epsilon)$ such that:

$$\int_{\partial D} k(r)|u(r)|^2 dr \leq \epsilon\|u\|^2 + C(\epsilon)\|u\|_H^2. \quad (2.5)$$

Moreover, we set $D(A) = \{u \in H^{2,2} : \text{curl } u = 0 \text{ on } \partial D\}$, and define the linear operator $A : D(A) \longrightarrow H$ as

$$Au = -\Delta u, \text{ i.e., } a(u, v) := (Au, v).$$

On the other hand, for all $u \in D(A)$ we have

$$(B(u, u), Au) = 0. \quad (2.6)$$

For $\beta > 0$ we will denote the β -power of the operator A by A^β and its domain by $D(A^\beta)$. Here $D(A^{-\beta})$ denotes the dual of $D(A^\beta)$. Note that for $k < 3/4$, we have $H^{k,2} = D(A^{k/2})$; the proof can be found in [11] Theorem 3.1. Set $\mathcal{H} = H^{1/2,2}$ and note that $\mathcal{H} = D(A^{1/4})$ and $V = D(A^{1/2})$. The continuous embedding $V \subset \mathcal{H} \subset H$ holds. Moreover, \mathcal{H} is an interpolation space, that is there exists a constant $a_0 > 0$ such that

$$\|u\|_{\mathcal{H}}^2 \leq a_0\|u\|_H\|u\|, \text{ for all } u \in V. \quad (2.7)$$

Since $\mathcal{H} \subset L^4(D)$ and $\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle$, we deduce

$$|\langle B(u, v), w \rangle| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|w\|, \quad (2.8)$$

and B can be extended as a bilinear operator from $\mathcal{H} \times \mathcal{H} \longrightarrow V'$.

In place of equations (2.1) we will consider the abstract stochastic evolution equation:

$$du(t) + \nu Au(t)dt + B(u(t), u(t))dt = \sigma(t, u(t))dW(t) \quad (2.9)$$

on the time interval $[0, T]$ with the initial condition $u(0) = \zeta$ and B satisfies conditions (2.2), (2.3), (2.6) and (2.8).

2.2. Stochastic driving force. Let Q be a linear positive operator in the Hilbert space H which is trace class, and hence compact. Let $H_0 = Q^{\frac{1}{2}}H$; then H_0 is a Hilbert space with the scalar product

$$(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi), \quad \forall \phi, \psi \in H_0,$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot, \cdot)_0}$. The embedding $i : H_0 \rightarrow H$ is Hilbert-Schmidt and hence compact, and moreover, $i^*i = Q$. Let $L_Q \equiv L_Q(H_0, H)$ be the space of linear operators $S : H_0 \rightarrow H$ such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from H to H . The norm in the space L_Q is defined by $|S|_{L_Q}^2 = \text{tr}(SQS^*)$, where S^* is the adjoint operator of S . The L_Q -norm can also be written in the form

$$|S|_{L_Q}^2 = \text{tr}([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k \geq 1} |SQ^{1/2}\psi_k|_H^2 = \sum_{k \geq 1} |[SQ^{1/2}]^*\psi_k|_H^2 \quad (2.10)$$

for any orthonormal basis (ψ_k) in H .

Let $(W(t), t \geq 0)$ be a Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, taking values in H and with covariance operator Q . This means that W is Gaussian, has independent time increments and that for $s, t \geq 0$, $f, g \in H$,

$$\mathbb{E}(W(s), f) = 0 \quad \text{and} \quad \mathbb{E}[(W(s), f)(W(t), g)] = (s \wedge t)(Qf, g).$$

Let (β_j) be standard (scalar) mutually independent Wiener processes, (e_j) be an orthonormal basis in H consisting of eigen-elements of Q , with $Qe_j = q_j e_j$. Then W has the following representation

$$W(t) = \lim_{n \rightarrow \infty} W_n(t) \quad \text{in } L^2(\Omega; H) \quad \text{with } W_n(t) = \sum_{1 \leq j \leq n} q_j^{1/2} \beta_j(t) e_j, \quad (2.11)$$

and $\text{Trace}(Q) = \sum_{j \geq 1} q_j$. For details concerning this Wiener process see e.g. [19].

Let $k \geq 0$, $q \in [2, \infty)$ and let $R(H_0, W^{k,q})$ denote the space of all γ -radonifying mappings from H_0 into $W^{k,q}$, which are analogues of Hilbert-Schmidt operators when the Hilbert Sobolev spaces $W^{k,2}$ are replaced by the more general Banach spaces $W^{k,q}$. The definitions and some basic properties of stochastic calculus in the framework of special Banach spaces, including the case of non-Hilbert Sobolev spaces, can be found in [11]; see also [7], [21], [36] and [37]. For the sake of self-completeness, they are described in sub-section 6.2 of the Appendix. The radonifying norm $\|S\|_{R(H_0, W^{k,q})}$ of an element S of $R(H_0, W^{k,q})$ is defined in (6.9); it is the extension of the L_Q norm of $S \in L_Q$ which is the particular case $k = 0$ and $q = 2$.

2.3. Assumptions. Given a viscosity coefficient $\nu > 0$, consider the following stochastic Navier-Stokes equations

$$du^\nu(t) + [\nu Au^\nu(t) + B(u^\nu(t), u^\nu(t))] dt = \sqrt{\nu} \sigma_\nu(t, u^\nu(t)) dW(t), \quad (2.12)$$

where the noise intensity $\sigma_\nu : [0, T] \times V \rightarrow L_Q(H_0, H)$ of the stochastic perturbation is properly normalized by the square root of the viscosity coefficient ν . We assume that σ_ν satisfies the following growth and Lipschitz conditions:

Condition (C1): For every $\nu > 0$, $\sigma_\nu \in \mathcal{C}([0, T] \times V; L_Q(H_0, H))$, there exist constants $K_i, L_1 \geq 0$ such that for every $t \in [0, T]$, $\nu > 0$ and $u, v \in V$:

- (i) $|\sigma_\nu(t, u)|_{L_Q}^2 \leq K_0 + K_1 |u|_H^2$,
- (ii) $|\sigma_\nu(t, u) - \sigma_\nu(t, v)|_{L_Q}^2 \leq L_1 |u - v|_H^2$.

For technical reasons, in order to prove a large deviation principle for the distribution of the solution to (2.12) as the viscosity coefficient ν converges to 0, we will need some precise estimates on the solution of the equation deduced from (2.12) by shifting the Brownian W by some random element of its RKHS. This cannot be deduced from similar ones on u by means of a Girsanov transformation; indeed, the Girsanov density is not uniformly bounded in $L^2(P)$ when the intensity of the noise tends to zero (see e.g. [23] or [18]).

To describe a set of admissible random shifts, we introduce the class \mathcal{A} as the set of H_0 -valued (\mathcal{F}_t) -predictable stochastic processes h such that $\int_0^T |h(s)|_0^2 ds < \infty$, a.s. For fixed $M > 0$, let

$$S_M = \left\{ h \in L^2(0, T; H_0) : \int_0^T |h(s)|_0^2 ds \leq M \right\}.$$

The set S_M , endowed with the following weak topology, is a Polish (complete separable metric) space (see e.g. [13]): $d_1(h, k) = \sum_{k \geq 1} \frac{1}{2^k} \left| \int_0^T (h(s) - k(s), \tilde{e}_k(s))_0 ds \right|$, where $(\tilde{e}_k(s), k \geq 1)$ is an orthonormal basis for $L^2(0, T; H_0)$. For $M > 0$ set

$$\mathcal{A}_M = \{h \in \mathcal{A} : h(\omega) \in S_M, \text{ a.s.}\}. \quad (2.13)$$

In order to define the stochastic controlled equation, we introduce for $\nu \geq 0$ a family of intensity coefficients $\tilde{\sigma}_\nu$ which act on a random element $h \in \mathcal{A}_M$ for some $M > 0$. The case $\nu = 0$ will be that of an inviscid limit "deterministic" equation with no stochastic integral, and which can be dealt with for fixed ω . We assume that for any $\nu \geq 0$ the coefficient $\tilde{\sigma}_\nu$ satisfies the following condition, similar to **(C1)** and weaker since the L_Q norm is replaced by the smaller one of $L(H, H_0)$.

Condition (C1Bis): For any $\nu \geq 0$, $\tilde{\sigma}_\nu \in \mathcal{C}([0, T] \times V; L(H_0, H))$ and there exist non negative constants \tilde{K}_i and \tilde{L}_1 such that for every $t \in [0, T]$, $\nu \geq 0$ and $u, v \in V$:

$$|\tilde{\sigma}_\nu(t, u)|_{L(H_0, H)} \leq \tilde{K}_0 + \tilde{K}_1 |u|_H, \quad (2.14)$$

$$|\tilde{\sigma}_\nu(t, u) - \tilde{\sigma}_\nu(t, v)|_{L(H_0, H)} \leq \tilde{L}_1 |u - v|_H. \quad (2.15)$$

Examples of coefficients σ_ν and $\tilde{\sigma}_\nu$ which satisfy conditions **(C1)** and **(C1Bis)**, of Nemytski form, are provided in subsection 6.3 of the Appendix.

Let $\nu > 0$, $M > 0$, $h \in \mathcal{A}_M$, ζ be an H -valued random variable independent of W . Under Conditions **(C1)** and **(C1Bis)**, we consider the nonlinear SPDE

$$\begin{aligned} du_h^\nu(t) + [\nu Au_h^\nu(t) + B(u_h^\nu(t), (u_h^\nu(t)))] dt &= \sqrt{\nu} \sigma_\nu(t, u_h^\nu(t)) dW(t) + \tilde{\sigma}_\nu(t, u_h^\nu(t)) h(t) dt, \\ u_h^\nu(0) &= \zeta. \end{aligned} \quad (2.16)$$

Well posedness of the above equation as well as apriori bounds of the solution to this equation in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ are known for fixed $\nu > 0$ when $u = 0$ on ∂D (see e.g. [39] and [18]). We will prove them uniformly in $\nu \in (0, \nu_0]$ for some small ν_0 under different boundary conditions.

Let us introduce the following conditions that we will use later in the paper. The following conditions **(C2)** and **(C2Bis)** will allow to improve apriori estimates on the p -th moment of the solution to the stochastic controlled equation (2.16) in V , uniformly in time and on a "small" viscosity coefficient ν . They will also yield the existence of a solution to the inviscid deterministic equation, that is of (2.16) when $\nu = 0$.

Condition (C2): For every $\nu > 0$, $\sigma_\nu \in \mathcal{C}([0, T] \times D(A); L_Q(H_0, V))$ and there exist non negative constants K_i, L_1 such that for every $t \in [0, T]$, $\nu > 0$ and $u, v \in D(A)$:

- (i) $|\text{curl } \sigma_\nu(u)|_{L_Q}^2 \leq K_0 + K_1 \|u\|_V^2$,
- (ii) $|A^{1/2} \sigma_\nu(t, u) - A^{1/2} \sigma_\nu(t, v)|_{L_Q}^2 \leq L_1 \|u - v\|_V^2$.

Condition (C2Bis): For every $\nu \geq 0$, $\tilde{\sigma}_\nu \in \mathcal{C}([0, T] \times D(A); L(H_0, V))$, there exist non negative constants \tilde{K}_i, \tilde{L}_1 , such that for every $t \in [0, T]$, $\nu \geq 0$ and $u, v \in D(A)$:

- (iii) $|\text{curl } \tilde{\sigma}_\nu(t, u)|_{L(H_0, H)} \leq \tilde{K}_0 + \tilde{K}_1 \|u\|_V$,
- (iv) $|A^{1/2} \tilde{\sigma}_\nu(t, u) - A^{1/2} \tilde{\sigma}_\nu(t, v)|_{L(H_0, H)} \leq \tilde{L}_1 \|u - v\|_V$.

Again, sub-section 6.3 of the Appendix provides examples of Nemytski operators which satisfy all the conditions above.

2.4. Well Posedness and a priori estimates. Let us mention in this section that the results used to obtain the well posedness of solutions are similar to known ones with different boundary conditions. However the apriori estimates are more involved since we are seeking estimates uniform in the parameter $\nu > 0$ which will be used later in Section 5 to let $\nu \rightarrow 0$. Note that the results in this section would still be valid under more general assumptions than those stated in Conditions **(C1)**-**(C2Bis)**, similar to that in [23] and [18]. The corresponding Nemitsky operators defining σ_ν and $\tilde{\sigma}_\nu$ could include some gradient of the solution multiplied by the square root of the viscosity coefficient. However, to focus on the main contribution of the present paper compared with previous related works, we prefer to keep simpler and more transparent assumptions on the diffusion coefficient σ_ν and an unrelated coefficient $\tilde{\sigma}_\nu$.

We at first recall that an (\mathcal{F}_t) -predictable stochastic process $u_h^\nu(t, \omega)$ is called a *weak solution* in $X \subset \mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ for the stochastic equation (2.16) on $[0, T]$ with initial condition ζ if $u_h^\nu \in X$ a.s., and satisfies a.s. the equality

$$\begin{aligned} (u_h^\nu(t), v) - (\zeta, v) + \int_0^t [\nu(u_h^\nu(s), Av) + \langle B(u_h^\nu(s), v), u_h^\nu(s) \rangle] ds \\ = \sqrt{\nu} \int_0^t (\sigma_\nu(s, u_h^\nu(s)) dW(s), v) + \int_0^t (\tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s), v) ds. \end{aligned} \quad (2.17)$$

for all $v \in \text{Dom}(A)$ and all $t \in [0, T]$. Note that this solution is a strong one in the probabilistic meaning, that is written in terms of stochastic integrals with respect to the given Brownian motion W .

Proposition 2.1. *Let $T > 0$, $(\sigma_\nu, \nu > 0)$ and $(\tilde{\sigma}_\nu, \nu > 0)$ satisfy conditions **(C1)** and **(C1Bis)** respectively and let the initial condition ζ be such that $\mathbb{E}|\zeta|_H^{2p} < \infty$ for some $p \geq 2$. Then for any $M > 0$ and $\nu_0 > 0$, there exist positive constants $C_1(p, M)$ and $\tilde{C}_1(M)$ (depending also on $T, \nu_0, K_i, \tilde{K}_i, i = 0, 1, 2$, such that for any $\nu \in (0, \nu_0]$ and any $h \in \mathcal{A}_M$, (2.16) has a unique weak solution in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ which satisfies the following apriori estimates:*

$$\sup_{0 < \nu \leq \nu_0} \sup_{h \in \mathcal{A}_M} \mathbb{E} \left(\sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p} \right) \leq C_1(p, M) [1 + \mathbb{E}|\zeta|_H^{2p}], \quad (2.18)$$

and

$$\sup_{0 < \nu \leq \nu_0} \sup_{h \in \mathcal{A}_M} \nu \int_0^T \mathbb{E} (\|u_h^\nu(s)\|^2 + \|u_h^\nu(s)\|_{\mathcal{H}}^4) ds \leq \tilde{C}_1(M) [1 + \mathbb{E}|\zeta|_H^4]. \quad (2.19)$$

Proof. The proof, which is quite classical, requires some Galerkin approximation of u_h^ν , say $u_h^{\nu, n}$, for which apriori estimates are proved uniformly in n . Note that in our situation, these apriori estimates have to be obtained uniformly in $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$. Using a subsequence of $(u_h^{\nu, n}, n \geq 1)$ which converges in the weak or the weak-star topologies of appropriate spaces, one can then prove that there exists a solution to (2.16) (see e.g [18] or [39]). The proof of the uniqueness is standard and omitted. To ease notation, we replace the Galerkin approximation by the limit process u_h^ν to obtain the required apriori estimates uniformly in $n \geq 1$ and in $\nu \in (0, \nu_0]$ for some $\nu_0 > 0$ under slightly more general boundary conditions; the proof can then be completed as in the appendix of [18]. If the well-posedness is already known, we use the solution u_h^ν instead of the Galerkin approximation. Let $\nu > 0$, $h \in \mathcal{A}_M$; for every $N > 0$, let $\tau_N = \inf \{t \geq 0, |u_h^\nu(t)|_H \geq N\} \wedge T$.

Applying Itô's formula first to $|\cdot|_H^2$ and the process $u_h^\nu(\cdot \wedge \tau_N)$, then to the map $x \mapsto x^p$ for $p \geq 2$ and the process $|u_h^\nu(\cdot \wedge \tau_N)|_H^2$, we deduce:

$$|u_h^\nu(t \wedge \tau_N)|_H^{2p} + \nu 2p \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \|u_h^\nu(s)\|^2 ds \leq |u_h^\nu(0)|_H^{2p} + J(t) + \sum_{i=1}^5 T_i(t), \quad (2.20)$$

where

$$\begin{aligned} J(t) &= 2p\sqrt{\nu} \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} (\sigma_\nu(s, u_h^\nu(s)) dW(s), u_h^\nu(s)), \\ T_1(t) &= 2p\nu \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \int_{\partial D} k(r) |u_h^\nu(r)|_H^2 dr ds, \\ T_2(t) &= 2p \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \langle B(u_h^\nu(s), u_h^\nu(s)), u_h^\nu(s) \rangle ds, \\ T_3(t) &= 2p \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} (\tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s), u_h^\nu(s)) ds, \end{aligned}$$

$$\begin{aligned}
T_4(t) &= \nu p \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} |\sigma_\nu(s, u_h^\nu(s))|_{L_Q}^2 ds, \\
T_5(t) &= 2\nu p(p-1) \int_0^{t \wedge \tau_N} |\sigma_\nu^*(s, u_h^\nu(s)) u_h^\nu(s)|_0^2 |u_h^\nu(s)|_H^{2(p-2)} ds.
\end{aligned}$$

The incompressibility condition (2.2) implies that $T_2(t) = 0$ for any $t \in [0, T]$. Using (2.5), we deduce that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$T_1(t) \leq 2\nu p \epsilon \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \|u_h^\nu(s)\|^2 ds + 2\nu p C(\epsilon) \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p} ds.$$

Since $h \in \mathcal{A}_M$, the growth condition (2.14), the Cauchy-Schwarz and Hölder inequalities imply:

$$\begin{aligned}
T_3(t) &\leq 2p \int_0^{t \wedge \tau_N} \left[\tilde{K}_0 + (\tilde{K}_0 + \tilde{K}_1) |u_h^\nu(s)|_H^{2p} \right] |h(s)|_0 ds \\
&\leq 2p \tilde{K}_0 \sqrt{MT} + 2p (\tilde{K}_0 + \tilde{K}_1) \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p} |h(s)|_0 ds.
\end{aligned}$$

Using the growth condition **(C1)**, we deduce for $\nu \in (0, \nu_0]$:

$$T_4(t) + T_5(t) \leq \nu p(2p-1) K_0 T + \nu p(2p-1) (K_0 + K_1) \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p} ds.$$

Thus, the Itô formula (2.20) and the previous upper estimates of $T_i(t)$, $i = 1, \dots, 5$, imply that for any $t \in [0, T]$, $\epsilon \in (0, 1)$,

$$\begin{aligned}
&|u_h^\nu(t \wedge \tau_N)|_H^{2p} + 2\nu p(1-\epsilon) \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \|u_h^\nu(s)\|^2 ds \\
&\leq \tilde{Z} + \int_0^t \tilde{\varphi}(s) |u(s \wedge \tau_N)|_H^{2p} ds + J(t),
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\tilde{Z} &= |\zeta|_H^{2p} + 2p \tilde{K}_0 \sqrt{MT} + p(2p-1) \nu K_0 T, \\
\tilde{\varphi}(s) &= p \left[2\nu C(\epsilon) + (2p-1) \nu (K_0 + K_1) + 2(\tilde{K}_0 + \tilde{K}_1) |h(s)|_0 \right].
\end{aligned}$$

For $t \in [0, T]$, set

$$X(t) := \sup_{0 \leq s \leq t} |u_h^\nu(s \wedge \tau_N)|_H^{2p}, \quad Y(t) := \int_0^{t \wedge \tau_N} |u_h^\nu(s)|_H^{2p-2} \|u_h^\nu(s)\|^2 ds, \quad \tilde{I}(t) := \sup_{0 \leq s \leq t} J(s).$$

Let $\varepsilon = \frac{1}{2}$, $\nu \in (0, \nu_0]$, $\lambda \in (0, 1)$ and $\tilde{\alpha} = (1-\lambda)\nu p$. With these notations, the inequality (2.21) yields

$$\lambda X(t) + (1-\lambda) |u_h^\nu(t \wedge \tau_N)|_H^{2p} + \tilde{\alpha} Y(t) \leq \tilde{Z} + \int_0^t \tilde{\varphi}(s) X(s) ds + \tilde{I}(t). \tag{2.22}$$

Furthermore, using the Burkholder-Davis-Gundy inequality, condition **(C1)**, then Cauchy-Schwarz's and Young's inequalities, we deduce that for any $\beta > 0$,

$$\begin{aligned}
\mathbb{E} \tilde{I}(t) &\leq 6\sqrt{\nu p} \mathbb{E} \left(X(t) \int_0^{t \wedge \tau_N} [K_0 + (K_0 + K_1) |u_h^\nu(s)|_H^{2p}] ds \right)^{1/2} \\
&\leq \tilde{\beta} \mathbb{E} X(t) + \tilde{\gamma} \mathbb{E} \int_0^t X(s) ds + \bar{C},
\end{aligned}$$

where $\tilde{\gamma} = \frac{9\nu p^2}{\beta}(K_0 + K_1)$ and $\bar{C} = \frac{9\nu p^2}{\beta}K_0T$. Let $\lambda = \frac{1}{2}$, $\varphi = 2\tilde{\varphi}$, $\alpha = 2\tilde{\alpha}$, $\beta = 2\tilde{\beta}$, $\gamma = 2\tilde{\gamma}$ and $I(t) = \tilde{I}(t)$. Then for $t \in [0, T]$, we have a.s.

$$X(t) + \alpha Y(t) \leq 2\tilde{Z} + \int_0^t \varphi(s)X(s)ds + I(t), \quad \mathbb{E}I(t) \leq \beta \mathbb{E}X(t) + \gamma \mathbb{E} \int_0^t X(s)ds + 2\bar{C}.$$

Furthermore, for $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$, one has a.s. $\int_0^T \varphi(s)ds \leq \Phi(M, \nu_0)$, where $\Phi(M, \nu_0) = 4p(\tilde{K}_0 + \tilde{K}_1)\sqrt{MT} + 2p\nu_0[2C(\frac{1}{2}) + (2p-1)(K_0 + K_1)]$. Let $\tilde{\beta} > 0$ be such that $4\tilde{\beta} \exp(\Phi(M, \nu_0)) \leq 1$. Then since $X(\cdot)$ is bounded by N , Lemma A.1 in [18] (see also Lemma 3.9 in [23]) implies that for $t \in [0, T]$, we have:

$$\mathbb{E}[X(t) + \alpha Y(t)] \leq C(\mathbb{E}|\zeta|_H^{2p}, \nu_0, M, T),$$

for some constant $C(\mathbb{E}|\zeta|_H^{2p}, \nu_0, M, T)$ which does not depend on N , $\nu \in (0, \nu_0]$, $h \in \mathcal{A}_M$ and on the step n of the Galerkin approximation. Since the right handside in the above equation does not depend on N , letting $N \rightarrow \infty$ we obtain that $\tau_N \rightarrow T$ a.s. Hence there exists a constant $C_1 := C_1(\mathbb{E}|\zeta|_H^{2p}, \nu_0, M, T)$ such that the Galerkin approximation $u_h^{n,\nu}$ of u_h^ν satisfies:

$$\sup_{n \geq 1} \mathbb{E} \left(\sup_{0 \leq t \leq T} |u_h^{n,\nu}(t)|_H^{2p} + \nu \int_0^T [\|u_h^{n,\nu}(t)\|_{\mathcal{H}}^4 + \|u_h^{n,\nu}(s)\|^2] ds \right) \leq C_1$$

for any $n, \nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$. The proof is completed using a classical argument (see e.g. the Appendix of [18] for details.) \square

Proposition 2.2. *Let the assumptions of Proposition 2.1 be satisfied for $p = 1$ or some $p \in [2, \infty)$. Moreover, assume that the initial condition ζ is such that $\mathbb{E}\|\zeta\|^{2p} < \infty$ and that $(\sigma_\nu, \nu > 0)$ and $(\tilde{\sigma}_\nu, \nu > 0)$ satisfy respectively conditions **(C2)** and **(C2Bis)**. Then given any $M > 0$, there exists a positive constant $C_2(p, M)$ such that for $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$, the solution to (2.16) satisfies:*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|u_h^\nu(t)\|^{2p} + \nu \int_0^T |Au_h^\nu(s)|_H^2 ds \right) \leq C_2(p, M)(1 + \mathbb{E}\|\zeta\|^{2p}). \quad (2.23)$$

Proof. Let $\xi_h^\nu = \text{curl } u_h^\nu$, then it is a classical result that u_h^ν is solution of the following elliptic problem (see e.g. [5] and the references therein),

$$\begin{cases} -\Delta u_h^\nu = \nabla^\perp \xi_h^\nu & \text{in } D, \\ u_h^\nu \cdot n = \xi_h^\nu = 0 & \text{on } \partial D, \end{cases} \quad (2.24)$$

where $\nabla^\perp = (D_2, -D_1)$. Using the equation (2.24), we get that

$$-(\Delta u_h^\nu, \Delta u_h^\nu) = (\nabla^\perp \xi_h^\nu, \Delta u_h^\nu) = -(\nabla^\perp \xi_h^\nu, \nabla^\perp \xi_h^\nu).$$

Hence

$$|\Delta u_h^\nu|_H^2 = |\nabla^\perp \xi_h^\nu|_H^2 = \|D_2 \xi_h^\nu\|_{L^2(D)}^2 + \|D_1 \xi_h^\nu\|_{L^2(D)}^2 = \|\nabla \xi_h^\nu\|_H^2.$$

Using (6.3) we see that the proof of (2.23) reduces to check that there exists a constant $C(M, T, K_i, \tilde{K}_i) := C_3$ such that for any $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$,

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\xi_h^\nu(t)|_H^{2p} + \nu \int_0^T \|\nabla \xi_h^\nu(s)\|_H^2 ds \right) \leq C_3(1 + \mathbb{E}|\text{curl } \zeta|_H^{2p}). \quad (2.25)$$

We at first prove this inequality for the Galerkin approximation of the solution ; a standard argument extends it to u_h^ν and hence ξ_h^ν . Fix $N > 0$ and set $\bar{\tau}_N = \inf\{t \geq 0 : |\xi_h^\nu(t)|_H \geq N\} \wedge T$. Applying the curl to the evolution equation (2.16) yields $\xi_h^\nu(0) = \text{curl } \zeta$ and

$$d\xi_h^\nu(t) + \nu A \xi_h^\nu(t) dt + \text{curl } B(u_h^\nu(t), u_h^\nu(t)) dt =$$

$$\sqrt{\nu} \operatorname{curl} \sigma_\nu(s, u_h^\nu(t)) dW(t) + \operatorname{curl} \tilde{\sigma}_\nu(s, u_h^\nu(t)) h(t) dt. \quad (2.26)$$

Recall that equation (6.7) with $q = 2$, implies $(\operatorname{curl} B(u_h^\nu, u_h^\nu), \xi_h^\nu) = 0$ for $u \in D(A)$. Using Itô's formula for the square of the H norm, and then for the map $x \rightarrow |x|_H^p$ with $p \in [2, \infty)$, we obtain for $t \in [0, T]$:

$$|\xi_h^\nu(s \wedge \bar{\tau}_N)|_H^{2p} + 2p\nu \int_0^{t \wedge \bar{\tau}_N} |\nabla \xi_h^\nu(s)|_H^2 |\xi_h^\nu(s)|_H^{2p-2} ds = |\operatorname{curl} \zeta|_H^{2p} + \bar{J}(t) + \sum_{i=1}^3 \bar{T}_i(t), \quad (2.27)$$

where

$$\begin{aligned} \bar{J}(t) &= 2p\sqrt{\nu} \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-2} (\operatorname{curl} \sigma_\nu(s, u_h^\nu(s)) dW(s), \xi_h^\nu(s)), \\ \bar{T}_1(t) &= 2p \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-2} (\operatorname{curl} \tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s), \xi_h^\nu(s)) ds, \\ \bar{T}_2(t) &= \nu p \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-2} |\operatorname{curl} \sigma_\nu(s, u_h^\nu(s))|_{L_Q}^2 ds, \\ \bar{T}_3(t) &= 2\nu p(p-1) \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-4} |(\operatorname{curl} \sigma_\nu(s, u_h^\nu(s)))^* \xi_h^\nu(s)|_{H_0}^2 ds. \end{aligned}$$

Using the Cauchy-Schwarz inequality, **(C2Bis)** and (6.3) with $q = 2$, we get that

$$\begin{aligned} \bar{T}_1(t) &\leq 2p \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-1} |\operatorname{curl} \tilde{\sigma}_\nu(s, u)|_{L(H_0, H)} |h(s)|_0 ds \\ &\leq 2p \int_0^{t \wedge \bar{\tau}_N} \left[\tilde{K}_0 + (\tilde{K}_0 + 2C\tilde{K}_1) |\xi_h^\nu(s)|_H^{2p} + \tilde{K}_1 |u_h^\nu(s)|_H |\xi_h^\nu(s)|_H^{2p-1} \right] |h(s)|_0 ds. \end{aligned}$$

Using Cauchy-Schwarz's, Hölder's and Young's inequalities, we deduce:

$$\bar{T}_1(t) \leq 2p\tilde{K}_0\sqrt{MT} + \tilde{K}_1^{2p}\sqrt{MT} \sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p} + \int_0^{t \wedge \bar{\tau}_N} \psi_1(s) |\xi_h^\nu(s)|_H^{2p} ds,$$

where $\psi_1(s) := 2p(\tilde{K}_0 + 2C\tilde{K}_1) + (2p-1)|h(s)|_0$.

Furthermore, $\bar{T}_3(t)$ can be upper estimated in terms of $\bar{T}_2(t)$ as follows:

$$\bar{T}_3(t) \leq 2\nu p(p-1) \int_0^{t \wedge \bar{\tau}_N} |\operatorname{curl} \sigma_\nu(u_h^\nu(s))|_{L_Q}^2 |\xi_h^\nu(s)|_H^{2p-2} ds = 2(p-1)\bar{T}_2(t).$$

Finally, condition **(C2)**, (6.3) with $q = 2$, Hölder's and Young's inequalities, we obtain for $\nu \in (0, 1]$:

$$\begin{aligned} \bar{T}_2(t) + \bar{T}_3(t) &\leq \nu p(2p-1) \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-2} \left[K_0 + K_1 (|u_h^\nu(s)|_H^2 + 4C^2 |\xi_h^\nu(s)|_H^2) \right] ds \\ &\leq \nu(2p-1)T \left[pK_0 + K_1 \sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p} \right] + \int_0^{t \wedge \bar{\tau}_N} \nu \psi_2 |\xi_h^\nu(s)|_H^{2p} ds, \end{aligned}$$

where $\psi_2 = (2p-1)[p(K_0 + 4K_1C^2) + (p-1)K_1]$. Let

$$\bar{X}(t) = \sup_{0 \leq s \leq t} |\xi_h^\nu(s \wedge \bar{\tau}_N)|_H^{2p}, \quad \bar{Y}(t) = \int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p-2} |\nabla \xi_h^\nu(s)|_H^2 ds.$$

Then for $\bar{\alpha} = 2\nu p(1 - \bar{\lambda})$, $\bar{I}(t) = \sup_{0 \leq s \leq t} |\bar{J}(s)|$, $h \in \mathcal{A}_M$ and

$$\begin{aligned} \bar{Z} &:= |\operatorname{curl} \zeta|_H^{2p} + 2p\tilde{K}_0\sqrt{MT} + \nu p(2p-1)TK_0 \\ &\quad + (\tilde{K}_1^{2p}\sqrt{MT} + \nu(2p-1)TK_1) \sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p}, \end{aligned}$$

equation (2.27) and the upper bounds of $\bar{T}_i(t)$ imply that for $t \in [0, T]$ and $\nu \in (0, \nu_0]$,

$$\bar{\lambda}\bar{X}(t) + \bar{\alpha}\bar{Y}(t) + (1 - \bar{\lambda})|\xi_h^\nu(s \wedge \bar{\tau}_N)|_H^{2p} \leq \bar{Z} + \int_0^t [\psi_1(s) + \nu\psi_2] \bar{X}(s) ds + \bar{I}(t), \quad (2.28)$$

The Davies inequality, condition **(C2)**, (6.3) for $q = 2$, Cauchy-Schwarz's, Hölder's and Young's inequalities imply that for any $\bar{\beta} > 0$,

$$\begin{aligned} \mathbb{E}\bar{I}(t) &\leq 6p\sqrt{\nu}\mathbb{E}\left(\int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{4p-2} |\operatorname{curl} \sigma_\nu(u_h^\nu(s))|_{L_Q}^2 ds\right)^{\frac{1}{2}} \\ &\leq \bar{\beta}\mathbb{E}\bar{X}(t) + \frac{9p^2\nu}{\bar{\beta}}K_0T + \frac{9p^2\nu}{\bar{\beta}}(K_0 + 4C^2K_1)\mathbb{E}\int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p} ds \\ &\quad + \frac{9p^2\nu}{\bar{\beta}}K_1\left(\mathbb{E}\int_0^{t \wedge \bar{\tau}_N} |\xi_h^\nu(s)|_H^{2p} ds\right)^{\frac{p-1}{p}}\left(T\mathbb{E}\sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p}\right)^{\frac{1}{p}} \\ &\leq \bar{\beta}\mathbb{E}\bar{X}(t) + \bar{\gamma}\mathbb{E}\int_0^{t \wedge \bar{\tau}_N} \bar{X}(s) ds + \tilde{Z}, \end{aligned} \quad (2.29)$$

where $\bar{\gamma} = \frac{9p^2\nu}{\bar{\beta}}[p(K_0 + 4C^2K_1) + (p-1)K_1]$ and $\tilde{Z} := \frac{9p\nu T}{\bar{\beta}}[pK_0 + K_1\mathbb{E}(\sup_{0 \leq s \leq T} |u_h^\nu(s)|_H^{2p})]$. Set $\bar{\lambda} = \frac{1}{2}$, $\varphi(s) = 2(\psi_1(s) + \nu\psi_2)$, $\alpha = 2\bar{\alpha}$, $\beta = 2\bar{\beta}$, $\gamma = 2\bar{\gamma}$, and $I(t) = 2\bar{I}(t)$. Then for $\nu \in (0, \nu_0]$, $h \in \mathcal{A}_M$ and $t \in [0, T]$, we have:

$$\begin{aligned} \bar{X}(t) + \alpha\bar{Y}(t) &\leq \int_0^t \varphi(s)\bar{X}(s) ds + I(t) + 2\bar{Z}(t), \\ \mathbb{E}I(t) &\leq \beta\mathbb{E}\bar{X}(t) + \gamma\mathbb{E}\int_0^t \bar{X}(s) ds + \delta\mathbb{E}\bar{Y}(t) + 2\tilde{Z}. \end{aligned}$$

Furthermore, there exists a constant $\Psi(M, \nu_0) > 0$ such that almost surely, $\int_0^T \varphi(s) ds \leq \Psi(M, \nu_0)$ for $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$. Let $\bar{\beta} > 0$ be such that $4\bar{\beta}\exp(\Psi(M, \nu_0)) \leq 1$. Applying Lemma A.1 in [18] we deduce that for every $h \in \mathcal{A}_M$ and $\nu \in (0, \nu_0]$:

$$\sup_N \mathbb{E}\left(\sup_{0 \leq t \leq T} |\xi_h^\nu(t \wedge \bar{\tau}_N)|_H^{2p} + \nu \int_0^{\bar{\tau}_N} |\nabla \xi_h^\nu(s)|_H^2 ds\right) \leq C(\mathbb{E}|\operatorname{curl} \zeta|_H^{2p}, M, T).$$

Since the previous upper bound is uniform in N , we deduce that $\bar{\tau}_N \rightarrow T$ as $N \rightarrow \infty$. Thus, using the monotone convergence we get (2.25), which concludes the proof. \square

The following well-posedness result for problem (2.16) follows from Propositions 2.1 and 2.2; the proof is not given and we refer to [18] and [39] for details.

Theorem 2.3. *Assume that $(\sigma_\nu, \nu > 0)$ satisfies conditions **(C1)** and **(C2)** and $(\tilde{\sigma}_\nu, \nu > 0)$ satisfies conditions **(C1Bis)** and **(C2Bis)**. Let $p \in [2, \infty)$ be such that $\mathbb{E}(\|\zeta\|^{2p}) < \infty$. Then for every $M > 0$, $h \in \mathcal{A}_M$ and $\nu \in (0, \nu_0]$, there exists a unique weak solution u_h^ν in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ of equation (2.16) with initial condition $u_h^\nu(0) = \zeta \in V$. Furthermore, a.s. $u_h^\nu \in \mathcal{C}([0, T]; V)$ and the inequalities (2.18), (2.19) and (2.23) hold.*

3. WELL POSEDNESS OF THE INVISCID PROBLEM

The aim of this section is to deal with the inviscid case $\nu = 0$, that is with the Euler evolution equation

$$du_h^0(t) + B(u_h^0(t), u_h^0(t)) dt = \tilde{\sigma}_0(t, u_h^0(t)) h(t) dt, \quad u_h^0(0) = \zeta \quad (3.1)$$

in $[0, T] \times D$.

Theorem 3.1. *Let us assume that $\zeta \in V$ and that $\tilde{\sigma}_0$ satisfies conditions **(C1Bis)** and **(C2Bis)**. Then for all $M > 0$, $h \in \mathcal{A}_M$ and $T > 0$, there exists a.s. a solution $u_h^0 \in C([0, T]; H) \cap L^\infty(0, T; V)$ for the equation (3.1) with the initial condition $u_h^0 = \zeta$, such that for all $\varphi \in V$ and $t \in [0, T]$*

$$(u_h^0(t), \varphi) - \int_0^t \langle B(u_h^0(s), \varphi), u_h^0(s) \rangle ds = \int_0^t (\tilde{\sigma}_0(s, u_h^0(s)) h(s), \varphi) ds, \quad \text{a.s.} \quad (3.2)$$

Moreover, there exists a positive constant $C_3(M)$ (which also depends on \tilde{K}_0 , \tilde{K}_1 and T) such that for every $h \in \mathcal{A}_M$, one has a.s.

$$\sup_{0 \leq t \leq T} \|u_h^0(t)\| \leq C_3(M)(1 + \|\zeta\|). \quad (3.3)$$

Proof. For $\mu > 0$, let us approximate equation (3.1) by the solution $u_h^{0\mu}$ to the following Navier Stokes evolution equation:

$$du_h^{0\mu}(t) + [\mu A u_h^{0\mu}(t) + B(u_h^{0\mu}(t), u_h^{0\mu}(t))] dt = \tilde{\sigma}_0(t, u_h^{0\mu}(t)) h(t) dt, \quad u_h^{0\mu}(0) = \zeta, \quad (3.4)$$

with the same incompressibility and boundary conditions. If $\zeta \in H$, $\tilde{\sigma}_0$ satisfies the condition **(C1Bis)** and $h \in \mathcal{A}_M$ for $M > 0$, then Proposition 2.1 shows that a.s. equation (3.4) has a unique solution $u_h^{0\mu} \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ (see also [39] or [18]). Moreover, if $\zeta \in V$ and $\tilde{\sigma}_0$ satisfies **(C2Bis)**, then Proposition 2.2 implies that a.s. $u_h^{0\mu} \in \mathcal{C}([0, T]; V)$. In order to prove the existence of solutions for equation (3.1), we need some estimates on $u_h^{0\mu}$ uniform in $\mu > 0$. Multiply the equation (3.4) by $2u_h^{0\mu}$ and integrate over $[0, t] \times D$; then an argument similar to that used to prove Proposition 2.1, based on (6.5), the Cauchy-Schwarz and Young inequalities and assumption **(C1Bis)**, yields for every $\mu > 0$:

$$\begin{aligned} |u_h^{0\mu}(t)|_H^2 + 2\mu \int_0^t \|u_h^{0\mu}(s)\|^2 ds &\leq |\zeta|_H^2 + 2 \int_0^t (\tilde{\sigma}_0(s, u_h^{0\mu}(s)) h(s), u_h^{0\mu}(s)) ds \\ &\leq |\zeta|_H^2 + 2\tilde{K}_0 \sqrt{MT} + 2(\tilde{K}_0 + \tilde{K}_1) \int_0^t |u_h^{0\mu}(s)|_H^2 |h(s)|_0 ds. \end{aligned}$$

Hence, by Gronwall's lemma, we deduce the existence of a constant \tilde{C}_1 which depends on M, T, \tilde{K}_0 and \tilde{K}_1 such that:

$$\sup_{\mu > 0} \sup_{0 \leq t \leq T} |u_h^{0\mu}(t)|_H^2 \leq \tilde{C}_1(1 + |\zeta|_H^2). \quad (3.5)$$

Let $\xi_h^{0\mu}(t) := \text{curl } u_h^{0\mu}(t)$; then applying the curl operator to equation (3.4) and using (6.6) we obtain the following evolution equation

$$d\xi_h^{0\mu}(t) + \mu A \xi_h^{0\mu}(t) + B(u_h^{0\mu}(t), \xi_h^{0\mu}(t)) dt = \text{curl } \tilde{\sigma}_0(t, u_h^{0\mu}(t)) h(t) dt, \quad (3.6)$$

with the initial condition $\xi_h^{0\mu}(0) = \text{curl } \zeta$. Multiply the equation (3.6) by $2\xi_h^{0\mu}$ and integrate over $[0, T] \times D$ and use an argument similar to that in the proof of Proposition 2.2; since $\tilde{\sigma}_0$ satisfies the condition **(C2Bis)**, using (6.7) for $q = 2$, (6.3), Cauchy-Schwarz's and Young's inequalities, we deduce

$$\begin{aligned} |\xi_h^{0\mu}(t)|_H^2 + 2\mu \int_0^t \|\xi_h^{0\mu}(s)\|^2 ds &\leq |\text{curl } \zeta|_H^2 + 2 \int_0^t |\text{curl } \tilde{\sigma}_0(s, u_h^{0\mu}(s))|_{L(H_0, H)} |h(s)|_0 |\xi_h^{0\mu}(s)|_H ds \\ &\leq \|\zeta\|^2 + \left(2\tilde{K}_0 + \tilde{K}_1 \sup_{s \in [0, T]} |u_h^{0\mu}(s)|_H^2\right) \sqrt{MT} + 2 \int_0^t \left(\tilde{K}_0 + (2C + 1)\tilde{K}_1\right) |h(s)|_0 |\xi_h^{0\mu}(s)|_H^2 ds. \end{aligned}$$

Thus, (3.5) and Gronwall's lemma yield the existence of a constant $\tilde{C}_2 := \tilde{C}_2(M, T, \tilde{K}_0, \tilde{K}_1)$ such that for every $h \in \mathcal{A}_M$: $\sup_{\mu > 0} \sup_{0 \leq t \leq T} |\xi_h^{0\mu}(t)|_H^2 \leq \tilde{C}_2(1 + \|\zeta\|^2)$ a.s. Combining

this estimate, (3.5) and (6.3), we deduce the existence of a constant \tilde{C}_3 depending on M, T, \tilde{K}_0 and \tilde{K}_1 such that for any $h \in \mathcal{A}_M$ one has:

$$\sup_{\mu > 0} \sup_{0 \leq t \leq T} \|u_h^{0\mu}\| \leq \tilde{C}_3(1 + \|\zeta\|) \quad \text{a.s.} \quad (3.7)$$

Furthermore, we have $u_h^{0\mu} \in \mathcal{C}([0, T]; H) \cap L^\infty(0, T; V)$ a.s. for every $\mu > 0$, and

$$u_h^{0\mu}(t) = \zeta - \mu \int_0^t A u_h^{0\mu}(s) - \int_0^t B(u_h^{0\mu}(s), u_h^{0\mu}(s)) ds + \int_0^t \tilde{\sigma}_0(s, u_h^{0\mu}(s)) h(s) ds.$$

Using the estimates (3.5), (3.7), assumptions **(C2Bis)** on $\tilde{\sigma}_0$ and (6.4) for $q = 2$ and $r = 1$, we deduce the existence of a constant \tilde{C}_4 depending on M, T, \tilde{K}_0 and \tilde{K}_1 such that the following estimate holds for any $\mu \in (0, 1]$ and $h \in \mathcal{A}_M$:

$$\|u_h^{0\mu}\|_{W^{1,2}(0,T;V')} \leq \tilde{C}_4(1 + \|\zeta\|). \quad \text{a.s.} \quad (3.8)$$

By classical compactness arguments, we can extract a subsequence (still denoted $u_h^{0\mu}$) and prove the existence of a function $v \in W^{1,2}(0, T; V') \cap L^\infty(0, T; V)$ such that as $\mu \rightarrow 0$:

$$\begin{aligned} u_h^{0\mu} &\rightarrow v \quad \text{weakly in } L^2(0, T; V) \text{ and in } W^{1,2}(0, T; V'), \\ u_h^{0\mu} &\rightarrow v \quad \text{strongly in } L^2(0, T; H), \\ u_h^{0\mu} &\rightarrow v \quad \text{in the weak star topology of } L^\infty(0, T; V). \end{aligned}$$

Letting $\mu \rightarrow 0$ in equation (3.4), we deduce that the above limit v is solution of the equation (3.1), that is $v = u_h^0$. Moreover, (3.7) being uniform in $\mu > 0$, we deduce (3.3). \square

Uniqueness of the solution to the Euler equation is known to be a more difficult problem and the classical deterministic results use non-Hilbert Sobolev spaces $H^{1,q}$ for $q \in [2, +\infty)$. This requires to impose some $H^{1,q}$ -control on the coefficient $\tilde{\sigma}_0$, which is stated below in Condition **(C3qBis)** for $\nu = 0$. It will enable us to prove the uniqueness of the solution to the "deterministic" inviscid equation in $H^{1,q}$ when $2 \leq q < \infty$ in the next result. The following general assumptions **(C3q)** and **(C3qBis)** on σ_ν and $\tilde{\sigma}_\nu$ will also yield some apriori estimates for the q -th moment of the $H^{1,q}$ -norm of the solution to the stochastic controlled equation which will be proven in section 4. This will be needed in order to prove the large deviations result as $\nu \rightarrow 0$ in section 5.

Condition (C3q): Let $q \in [2, \infty)$; $\sigma_\nu \in \mathcal{C}([0, T] \times H^{2,q}; R(H_0, H^{1,q}))$ for $\nu > 0$, there exist non negative constants K_i such that for every $u \in H \cap H^{2,q}$ and $\nu > 0$, if $\xi = \text{curl } u$, $\|\text{curl } \sigma_\nu(t, u)\|_{R(H_0, L^q)}^2 \leq K_3 + K_4\|u\|_q^2 + K_5\|\xi\|_q^2$.

Condition (C3qBis): Let $q \in [2, \infty)$; $\tilde{\sigma}_\nu \in \mathcal{C}([0, T] \times H^{1,q}; L(H_0, H^{1,q}))$ for $\nu \geq 0$, and there exist non negative constants \tilde{K}_i such that for every $u \in H^{1,q}$ and $\nu > 0$ (resp. $u \in H^{2,q}$ for $\nu = 0$) if $\xi = \text{curl } u$, $\|\text{curl } \tilde{\sigma}_\nu(t, u)\|_{L(H_0, L^q)} \leq \tilde{K}_3 + \tilde{K}_4\|u\|_q + \tilde{K}_5\|\xi\|_q$.

The following theorem shows that if $\text{curl } \zeta$ is bounded, then the solution to (3.1) is unique. This will be a key ingredient of the identification for the rate function of the LDP in section 5.

Theorem 3.2. Let us assume that the assumptions of Theorem 3.1 are satisfied. Moreover, let us assume that $\text{curl } \zeta \in (L^\infty(D))^2$ and that condition **(C3qBis)** holds for every $q \in [2, \infty)$ and $\nu = 0$. Then, for every $M > 0$ and $h \in \mathcal{A}_M$, the solution of equation (3.1) with the initial condition $u_h^0 = \zeta$ is a.s. unique in $\mathcal{C}([0, T]; H) \cap L^\infty(0, T; H^{1,q})$ for every $q \in [2, \infty)$ and every $T > 0$. Moreover, there exist positive constants $C_4(M)$ and $\tilde{C}_4(M)$

(which also depend on T , \tilde{K}_i and $\|\zeta\|_{L^\infty(D)^2}$), such that for every $h \in \mathcal{A}_M$ and $q \in [2, \infty)$, one has a.s.

$$\sup_{0 \leq t \leq T} \|\operatorname{curl} u_h^0(t)\|_q \leq C_4(M)(1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_q), \quad (3.9)$$

$$\sup_{0 \leq t \leq T} \|\nabla u_h^0(t)\|_q \leq \bar{C}_4(M)q(1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_q). \quad (3.10)$$

Proof. The first step of the proof will establish the estimates (3.9) and (3.10). The second step will prove the uniqueness of the solution u_h^0 .

Step1. (Existence) Using (6.3) one sees that the proof of (3.10) reduces to that of (3.9), that is to check $L^q(D)$ upper bounds for $\xi_h^0(t) := \operatorname{curl} u_h^0(t)$. Replacing u_h^0 by its Galerkin approximation $u_{h,n}^0$, we may assume that $u_{h,n}^0 \in H^{2,q}$ and deduce the desired inequality by proving upper bounds which do not depend on n . To ease notations in the sequel, we skip the index n .

Let us apply the curl to the equation (3.1); the identity (6.6) yields

$$d\xi_h^0(t) + B(u_h^0(t), \xi_h^0(t)) dt = \operatorname{curl} \tilde{\sigma}_0(t, u_h^0(t)) h(t) dt, \quad \xi_h^0(0) = \operatorname{curl} \zeta. \quad (3.11)$$

Let us multiply the equation (3.11) by $q|\xi_h^0(t)|^{q-2}\xi_h^0(t)$ and integrate over $[0, t] \times D$; we obtain

$$\begin{aligned} \|\xi_h^0(t)\|_q^q + q \int_0^t \int_D (u_h^0(s) \cdot \nabla) \xi_h^0(s) |\xi_h^0(s)|^{q-2} \xi_h^0(s) dx ds &= \|\operatorname{curl} \zeta\|_q^q \\ &+ q \int_0^t \int_D \operatorname{curl} \tilde{\sigma}_0(s, u_h^0(s)) h(s) |\xi_h^0(s)|^{q-2} \xi_h^0(s) dx ds. \end{aligned}$$

Since $\xi_h^0(t) = \operatorname{curl} u_h^0(t)$, (6.7) implies $\int_D (u_h^0(s) \cdot \nabla) \xi_h^0(s) |\xi_h^0(s)|^{p-2} \xi_h^0(s) dx = 0$ for every s . On the other side, the Hölder and Young inequalities and **(C3qBis)** yield:

$$\begin{aligned} \|\xi_h^0(t)\|_q^q &\leq \|\operatorname{curl} \zeta\|_q^q + q \int_0^t (\tilde{K}_3 + \tilde{K}_4 \|u_h^0(s)\|_q + \tilde{K}_5 \|\xi_h^0(s)\|_q) |h(s)|_0 \|\xi_h^0(s)\|_q^{q-1} ds \\ &\leq \|\operatorname{curl} \zeta\|_q^q + \left(q\tilde{K}_3 + \tilde{K}_4 \sup_{0 \leq s \leq T} \|u_h^0(s)\|_q \right) \sqrt{MT} \\ &\quad + q(\tilde{K}_3 + \tilde{K}_4 + \tilde{K}_5) \int_0^t |h(s)|_0 \|\xi_h^0(s)\|_q^q ds. \end{aligned}$$

Finally, the inclusion $V = H^{1,2} \subset L^q(D)$ given by (6.1), the control of the V norm proven in (3.3) (which clearly also holds for the Galerkin approximation $u_{h,n}^0$ with an upper bound which does not depend on n) and Gronwall's lemma imply the existence of a non negative constant $\tilde{C}_5(M)$, depending on T, M, \tilde{K}_i such that for any $n \geq 1$ and $h \in \mathcal{A}_M$, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\xi_{h,n}^0(t)\|_q^q &\leq \left(\|\operatorname{curl} \zeta\|_q^q + \left[q\tilde{K}_3 + \tilde{K}_4 \sup_{0 \leq t \leq T} \|u_h^0(t)\|_q \right] \right) e^{q\tilde{C}_5(M)} \\ &\leq \left(\|\operatorname{curl} \zeta\|_q^q + [q\tilde{K}_3 + \tilde{K}_4 C(q)^q C_3(M)^q 2^{q-1} (1 + \|\zeta\|_q)] \right) e^{q\tilde{C}_5(M)}. \end{aligned} \quad (3.12)$$

Since $\sup\{q^{\frac{1}{q}} : 2 \leq q < \infty\} < \infty$, if $C(q)$ denotes the constant in (6.1), as $n \rightarrow \infty$ classical arguments conclude that $\sup_{0 \leq t \leq T} \|\xi_{h,n}^0(t)\|_q \leq \|\operatorname{curl} \zeta\|_q + C(T, M)(1 + C(q))(1 + \|\zeta\|)$ for every $h \in \mathcal{A}_M$ and $n \geq 1$. Using (6.3) for some $q_0 \in [2, q)$, we deduce that $\sup_{0 \leq t \leq T} \|\nabla u_{h,n}^0(t)\|_{q_0} \leq C(T, M, q_0)(1 + \|\operatorname{curl} \zeta\|_{q_0} + \|\zeta\|)$ a.s. Thus, the Sobolev embedding (6.2) yields the existence of a constant $\tilde{C}_6(M, T)$, such that $\sup_{0 \leq t \leq T} \|u_{h,n}^0(t)\|_{L^\infty(D)} \leq \tilde{C}_6(M, T)(1 + \|\operatorname{curl} \zeta\|_{q_0} + \|\zeta\|)$ a.s. for any $n \geq 1$ and $h \in \mathcal{A}_M$. Since D is bounded, using this inequality in (3.12), we deduce that $\sup_{0 \leq t \leq T} \|\xi_{h,n}^0(t)\|_q^q \leq \exp(q\tilde{C}_5(M)) [\|\operatorname{curl} \zeta\|_q^q +$

$3^q \bar{C}_6(M, T)^q (1 + \|\zeta\|^q + \|\operatorname{curl} \zeta\|_{q_0}^q)$ a.s. for every integer $n \geq 1$ and any $h \in \mathcal{A}_M$. Since $q \in [q_0, +\infty)$ and D is bounded, we deduce $\|\operatorname{curl} \zeta\|_{q_0}^q \leq [\lambda(D) \vee 1]^q \|\operatorname{curl} \zeta\|_q^q$. Letting $n \rightarrow \infty$ and using classical arguments, we conclude the proof of (3.10).

Step 2. (Uniqueness) Let us mention that the proof of the uniqueness is based on [2] and [40] adapted to the nonhomogeneous random case. Using the estimate (3.10) for some $q > 2$ (such as $q = 4$) and (6.2), we deduce that any solution u_h^0 to (3.1) belongs to $L^\infty((0, T) \times D)$. Let u_h^0 and v_h^0 be two solutions for equation (3.1) with the same initial condition and let us denote by $z := u_h^0 - v_h^0$; then z is solution of $z(0) = 0$ and

$$dz(s) + [B(u_h^0(s), u_h^0(s)) - B(v_h^0(s), v_h^0(s))] ds = [\tilde{\sigma}_0(s, u_h^0(s)) - \tilde{\sigma}_0(s, v_h^0(s))] h(s) ds.$$

Let us multiply the above equation by $z(t)$ and integrate on D , use assumption **(C1Bis)** on $\tilde{\sigma}_0$, the Schwarz and Hölder inequalities and (3.10). This yields for any $q \in (1, \infty)$, when $q^* = \frac{q}{q-1}$ denotes the conjugate exponent of q :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |z(t)|_H^2 &= -(B(z(t), u_h^0(t)), z(t)) + ([\tilde{\sigma}_0(t, u_h^0(t)) - \tilde{\sigma}_0(t, v_h^0(t))] h(t), z(t)) \\ &\leq \int_D |z(t)|^2(x) |\nabla u_h^0(t)|(x) dx + |(\tilde{\sigma}_0(t, u_h^0(t)) - \tilde{\sigma}_0(t, v_h^0(t)))|_{L(H_0, H)} |h(t)|_0 |z(t)|_H \\ &\leq \|\nabla u_h^0(t)\|_q \|z(t)\|_{L^\infty(D)}^{\frac{2}{q}} |z(t)|_H^{\frac{2}{q^*}} + \tilde{L}_1 |u_h^0(t) - v_h^0(t)|_H |h(t)|_0 |z(t)|_H. \end{aligned}$$

Set $Z := \sup_{0 \leq t \leq T} \|z(t)\|_{L^\infty(D)}$ and $X(t) := |z(t)|_H^2$. Since D is bounded, there exists a constant $C \geq 1$ such that $\|\operatorname{curl} \zeta\|_q \leq C \|\operatorname{curl} \zeta\|_\infty$ for every $q \in [2, \infty)$; then $X(0) = 0$ and for $t \in [0, T]$, (3.10) yields

$$X'(t) \leq 2Cq \bar{C}_4(M) [1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_{L^\infty(D)}] Z^{\frac{2}{q}} X(t)^{1-\frac{1}{q}} + 2\tilde{L}_1 |h(t)|_0 X(t),$$

which leads to

$$\int_0^t \frac{X'(s)}{X(s)^{1-1/q}} ds \leq 2Cq \bar{C}_4(M) [1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_{L^\infty(D)}] Z^{\frac{2}{q}} t + \int_0^t 2\tilde{L}_1 |h(s)|_0 X(s)^{1/q} ds.$$

Hence, using Gronwall's lemma, we deduce that for $q \in [2, \infty)$ and $t \in [0, T]$,

$$\begin{aligned} X(t)^{\frac{1}{q}} &\leq 2\bar{C}_4(M) [1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_{L^\infty(D)}] Z^{\frac{2}{q}} t + \frac{2}{q} \int_0^t \tilde{L}_1 |h(s)|_0 X(s)^{\frac{1}{q}} ds \\ &\leq 2\bar{C}_4(M) [1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_{L^\infty(D)}] Z^{\frac{2}{q}} t \exp(\tilde{L}_1 \sqrt{MT}). \end{aligned}$$

Finally, we get the following estimate for any $T^* \in [0, T]$ and $q \in (2, \infty)$:

$$\sup_{0 \leq t \leq T^*} |z(t)|_H^2 \leq \left(2\bar{C}_4(M) [1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_{L^\infty(D)}] T^* \exp(2\tilde{L}_1 \sqrt{MT}) \right)^q Z^2. \quad (3.13)$$

Thus, choosing $T_1^* > 0$ small enough and letting $q \rightarrow \infty$, we deduce that $|z(t)|_H^2 = 0$ for every $t \in [0, T_1^*]$. Repeating this argument with $u_h^0(T_1^*) = v_h^0(T_1^*)$ instead of ζ and using (6.2), (3.10) and (3.3), we conclude that there exists $T^* > 0$ such that $|z(t)|_H^2 = 0$ for every integer $k = 0, 1, \dots$ and any $t \in [T_1^* + kT^*, T_1^* + (k+1)T^*] \cap [0, T]$. This concludes the proof of the uniqueness. \square

4. APRIORI BOUNDS OF THE STOCHASTIC CONTROLLED EQUATION IN $H^{1,q}$

In order to prove the large deviation principle for the solution u to (2.1), we need to obtain more regularity and apriori bounds for the solution u_h^ν to the stochastic controlled equation (2.16) in the Sobolev spaces $H^{1,q}$ for $q \in [2, +\infty)$. This requires some more conditions on the diffusion coefficient σ_ν and $\tilde{\sigma}_\nu$ introduced in the previous section. It

also relies on the stochastic calculus in Banach spaces, which is briefly described in the subsection 6.2 of the Appendix.

Proposition 4.1. *Suppose that $\mathbb{E}|\zeta|_H^{2p} < \infty$ for some $p \in [2, \infty)$ and let $q \in [2, \infty)$ be such that $E\|\zeta\|_{H^{1,q}}^q < \infty$. Assume that σ_ν satisfies conditions **(C1)**–**(C3q)** and $\tilde{\sigma}_\nu$ satisfies conditions **(C1Bis)**–**(C3qBis)**. Then for $M > 0$, $\nu_0 > 0$, $h \in \mathcal{A}_M$ and $\nu \in (0, \nu_0]$, the solution u_h^ν to (2.16) belongs to $L^\infty(0, T; H^{1,q})$ a.s. Furthermore, there exists a constant $C_5(M, q)$ such that*

$$\sup_{0 < \nu \leq \nu_0} \sup_{h \in \mathcal{A}_M} \mathbb{E} \left(\sup_{0 \leq t \leq T} \|u_h^\nu(t)\|_{H^{1,q}}^q \right) \leq C_5(M, q) (1 + \mathbb{E}\|\zeta\|_{H^{1,q}}^q). \quad (4.1)$$

Proof. The Sobolev embedding inequality (6.1) and Proposition 2.2 imply that for $0 < \nu \leq \nu_0$ and $h \in \mathcal{A}_M$, $\mathbb{E}(\sup_{0 \leq t \leq T} \|u_h^\nu(t)\|_q^q) \leq C(q)^q C_2(q, M)(1 + \mathbb{E}\|\zeta\|_q^q)$. Using the inequality (6.3), one sees that the proof of (4.1) reduces to check that if $\zeta_h^\nu = \text{curl } u_h^\nu$,

$$\sup_{0 < \nu \leq \nu_0} \sup_{h \in \mathcal{A}_M} \mathbb{E} \left(\sup_{0 \leq t \leq T} \|\xi_h^\nu(t)\|_q^q \right) \leq \bar{C}_6(M, q) (1 + \mathbb{E}\|\text{curl } \zeta\|_q^q). \quad (4.2)$$

We use once more the Galerkin approximation $u_{h,n}^\nu$ of u_h^ν and prove an estimate similar to (4.2) for $\xi_{h,n}^\nu = \text{curl } u_{h,n}^\nu$ with a constant $C_6(M, q)$ which does not depend on n . The process $\xi_{h,n}^\nu$ satisfies an equation similar to (2.26) and once more to ease notations, we will skip the index n . Let $\langle \cdot, \cdot \rangle$ denote the duality between $L^q(D)$ and $L^{q^*}(D)$ for some $q^* = \frac{q}{q-1}$. For fixed $N > 0$, let $\tau_N = \inf\{t \geq 0 : \|\xi_h^\nu(t)\|_q \geq N\} \wedge T$. The Itô formula (6.11) and the upper estimate (6.12) yield

$$\|\xi_h^\nu(t \wedge \tau_N)\|_q^q \leq \|\text{curl } \zeta\|_q^q + J(t) + \sum_{1 \leq i \leq 4} T_i(t), \quad (4.3)$$

where we have:

$$\begin{aligned} J(t) &= q\sqrt{\nu} \int_0^{t \wedge \tau_N} \langle |\xi_h^\nu(s)|^{q-2} \xi_h^\nu(s), \text{curl } \sigma_\nu(s, u_h^\nu(s)) dW(s) \rangle, \\ T_1(t) &= -q\nu \int_0^{t \wedge \tau_N} \langle |\xi_h^\nu(s)|^{q-2} \xi_h^\nu(s), A\xi_h^\nu(s) \rangle ds, \\ T_2(t) &= -q \int_0^{t \wedge \tau_N} \langle |\xi_h^\nu(s)|^{q-2} \xi_h^\nu(s), \text{curl } B(u_h^\nu(s), u_h^\nu(s)) \rangle ds, \\ T_3(t) &= q \int_0^{t \wedge \tau_N} \langle |\xi_h^\nu(s)|^{q-2} \xi_h^\nu(s), \text{curl } \tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s) \rangle ds, \\ T_4(t) &= \frac{q}{2}(q-1)\nu \int_0^{t \wedge \tau_N} \|\text{curl } \sigma_\nu(s, u_h^\nu(s))\|_{R(H_0, L^q)}^2 \|\xi_h^\nu(s)\|_q^{q-2} ds. \end{aligned}$$

Since $\xi_h^\nu = 0$ on ∂D and $A = -\Delta$, we have:

$$\begin{aligned} T_1(t) &= -q\nu \int_0^{t \wedge \tau_N} ds \int_D \langle \nabla(|\xi_h^\nu(s)|^{q-2} \xi_h^\nu(s)), \nabla \xi_h^\nu(s) \rangle dx \\ &= -q(q-1)\nu \int_0^{t \wedge \tau_N} ds \int_D |\xi_h^\nu(s)|^{q-2} |\nabla \xi_h^\nu(s)|^2 dx. \end{aligned}$$

Since $\xi_h^\nu = \text{curl } u_h^\nu$, equation (6.7) implies that $T_2(t) = 0$. Hölder's and Young's inequalities and the assumption **(C3qBis)** yield:

$$T_3(t) \leq q \int_0^{t \wedge \tau_N} \|\xi_h^\nu(s)\|_{q^*}^{q-1} \|\text{curl } \tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s)\|_q ds$$

$$\leq q\tilde{K}_3\sqrt{MT} + \tilde{K}_4 \int_0^{t\wedge\tau_N} \|u_h^\nu(s)\|_q^q |h(s)|_0 ds + \int_0^{t\wedge\tau_N} \|\xi_h^\nu(s)\|_q^q q[\tilde{K}_3 + \tilde{K}_4 + \tilde{K}_5] |h(s)|_0 ds.$$

Condition **(C3q)**, Hölder's and Young's inequalities imply that for any $\nu \in (0, \nu_0]$,

$$\begin{aligned} T_4(t) &\leq \frac{q(q-1)}{2} \nu \int_0^{t\wedge\tau_N} \|\xi_h^\nu(s)\|_q^{q-2} \left[K_3 + K_4 \|u_h^\nu(s)\|_q^2 + K_5 \|\xi_h^\nu(s)\|_q^2 \right] ds \\ &\leq \frac{q(q-1)}{2} \nu K_3 T + \nu(q-1) K_4 \int_0^{t\wedge\tau_N} \|u_h^\nu(s)\|_q^q ds \\ &\quad + \nu \frac{q-1}{2} \int_0^{t\wedge\tau_N} [q(K_3 + K_5) + K_4(q-2)] \|\xi_h^\nu(s)\|_q^q ds. \end{aligned}$$

For $t \in [0, T]$, let

$$X(t) = \sup_{0 \leq s \leq t} \|\xi_h^\nu(s \wedge \tau_N)\|_q^q \quad \text{and} \quad Y(t) = \int_0^{t\wedge\tau_N} ds \int_D |\xi_h^\nu(s)|^{q-2} |\nabla \xi_h^\nu(s)|^2 dx.$$

Then for any $\lambda \in (0, 1)$, the inequality (4.3) and the above estimates of $T_i(t)$ imply that

$$\lambda X(t) + (1-\lambda) \|\xi_h^\nu(t \wedge \tau_N)\|_q^q + \nu q(q-1)(1-\lambda)Y(t) \leq Z + \int_0^t \varphi(s)X(s)ds + I(t),$$

where

$$I(t) = \sup_{0 \leq s \leq t} J(s),$$

$$Z = \|\operatorname{curl} \zeta\|_q^q + q\tilde{K}_3\sqrt{MT} + \frac{q(q-1)}{2} \nu K_3 T + \int_0^{t\wedge\tau_N} [\nu K_4(q-1) + \tilde{K}_4 |h(s)|_0] \|u_h^\nu(s)\|_q^q ds,$$

$$\varphi(s) = q(\tilde{K}_3 + \tilde{K}_4 + \tilde{K}_5) |h(s)|_0 + \frac{q-1}{2} \nu q(K_3 + K_4 + K_5).$$

Set $\lambda = \frac{1}{2}$; then there exists a constant $\Phi(\nu_0, M)$ such that for $\nu \in (0, \nu_0]$ and $h \in \mathcal{A}_M$, almost surely let $\int_0^T \varphi(s)ds \leq \Phi(\nu_0, M)$ and

$$\frac{1}{2}X(t) + \frac{\nu}{2}q(q-1)Y(t) \leq Z + \int_0^t \varphi(s)X(s)ds + I(t).$$

Furthermore, using the Burkholder-Davies-Gundy inequality (6.10), condition **(C3q)**, Hölder's and Young's inequalities, we deduce that for any $\beta > 0$,

$$\begin{aligned} \mathbb{E}I(t) &\leq \sqrt{\nu} C_1 q \mathbf{E} \left(\int_0^{t\wedge\tau_N} \|\operatorname{curl} \sigma_\nu(s, u_h^\nu(s))\|_{R(H_0, L_q)}^2 \|\xi_h^\nu(s)\|_q^{2(q-1)} ds \right)^{\frac{1}{2}} \\ &\leq \sqrt{\nu} C_1 q \mathbb{E} \left(\sup_{0 \leq s \leq t} \|\xi_h^\nu(s \wedge \tau_N)\|_q^{\frac{q}{2}} \left[\int_0^{t\wedge\tau_N} \|\xi_h^\nu(s)\|_q^{q-2} \left\{ K_3 + K_4 \|u_h^\nu(s)\|_q^2 + K_5 \|\xi_h^\nu(s)\|_q^2 \right\} ds \right]^{\frac{1}{2}} \right) \\ &\leq \beta \mathbb{E}X(t) + \gamma \mathbf{E} \int_0^t X(s)ds + \bar{Z}, \end{aligned}$$

where

$$\gamma = \frac{1}{4\beta} \nu C_1^2 [q^2(K_3 + K_5) + qK_4(q-2)], \quad \bar{Z} = \frac{K_4}{2\beta} \nu q C_1^2 \mathbb{E} \int_0^{t\wedge\tau_N} \|u_h^\nu(s)\|_q^q ds + \frac{K_3 T \nu q^2}{4\beta}.$$

Set $\alpha = \frac{\nu}{2}q(q-1)$, and choose $\beta > 0$ such that $2\beta e^{\Phi(\nu_0, M)} \leq 1/2$. Then, using once more Lemma A1 in [18], we conclude that (4.2) holds for the Galerkin approximation $\xi_{h,n}^\nu$ of ξ_h^ν with a constant $C_6(M, q)$ which does not depend on n . A classical weak convergence argument concludes the proof. \square

5. LARGE DEVIATIONS

We will prove a large deviation principle using a weak convergence approach [12, 13], based on variational representations of infinite dimensional Wiener processes. For every $\nu > 0$, let $\sigma_\nu = \tilde{\sigma}_\nu$ satisfy the conditions **(C1)**, **(C2)** and **(C3q)** for every $q \in [2, +\infty)$. Furthermore, we assume that the following condition holds:

Condition (C4): There exists σ_0 satisfying conditions **(C1)**, **(C2)** and **(C3q)** for every $q \in [2, +\infty)$, such that for some map $\nu \in (0, +\infty) \rightarrow C(\nu) \in [0, +\infty)$ which converges to 0 as $\nu \rightarrow 0$, the upper estimate

$$\sup_{0 \leq t \leq T} |\sigma_\nu(t, u) - \sigma_0(t, u)|_{L(H_0, H)} \leq C(\nu) [1 + |u|_H] \quad (5.1)$$

holds for $u \in H$ and $\nu > 0$.

Note that as in the case of Hilbert-Schmidt operators with Hilbert spaces, we have $\|\Phi\|_{L(H_0, L^q)} \leq C\|\Phi\|_{R(H_0, L^q)}$. Then, for $\nu \geq 0$, the coefficients σ_ν also satisfy the conditions **(C1Bis)**-**(C3qBis)** with appropriate coefficients.

Let \mathcal{B} denote the Borel σ -field of the Polish space

$$\mathcal{X} = C([0, T]; H) \cap L^\infty(0, T; H^{1,q} \cap V) \cap L^2(0, T; \mathcal{H}) \quad (5.2)$$

endowed with the norm $\|u\|_{\mathcal{X}} := \left(\int_0^T \|u(t)\|_{\mathcal{H}}^2 dt \right)^{1/2}$ and

$$\mathcal{Y} = \{\zeta \in V, \text{ such that } \text{curl } \zeta \in L^\infty(D)\} \quad (5.3)$$

endowed with the norm $\|\cdot\|_{\mathcal{Y}}$ defined by:

$$\|\zeta\|_{\mathcal{Y}}^2 := \|\zeta\|^2 + \|\text{curl } \zeta\|_{L^\infty}^2.$$

Note that using (6.3) and (6.1) we deduce that $\mathcal{Y} \subset H^{1,q}$ for any $q \in [2, \infty)$. We will establish a LDP in the set \mathcal{X} for the family of distributions of the solutions $u^\nu = \mathcal{G}_\zeta^\nu(\sqrt{\nu}W)$ to the evolution equation (2.12) with initial condition $u^\nu(0) = \zeta \in \mathcal{Y}$.

Definition 5.1. *The random family (u^ν) is said to satisfy a large deviation principle on \mathcal{X} with the good rate function I if the following conditions hold:*

***I is a good rate function.** The function $I : \mathcal{X} \rightarrow [0, \infty]$ is such that for each $M \in [0, \infty[$ the level set $\{\phi \in \mathcal{X} : I(\phi) \leq M\}$ is a compact subset of \mathcal{X} .*

For $A \in \mathcal{B}$, set $I(A) = \inf_{u \in A} I(u)$.

Large deviation upper bound. *For each closed subset F of \mathcal{X} :*

$$\limsup_{\nu \rightarrow 0} \nu \log \mathbb{P}(u^\nu \in F) \leq -I(F).$$

Large deviation lower bound. *For each open subset G of \mathcal{X} :*

$$\liminf_{\nu \rightarrow 0} \nu \log \mathbb{P}(u^\nu \in G) \geq -I(G).$$

Let $\mathcal{C}_0 = \{\int_0^\cdot h(s)ds : h \in L^2(0, T; H_0)\} \subset \mathcal{C}([0, T]; H_0)$. Given $\zeta \in \mathcal{Y}$ define $\mathcal{G}_\zeta^0 : \mathcal{C}([0, T]; H_0) \rightarrow \mathcal{X}$ by $\mathcal{G}_\zeta^0(g) = u_h^0$ where $g = \int_0^\cdot h(s)ds \in \mathcal{C}_0$ and u_h^0 is the solution to the (inviscid) control equation (3.1) with initial condition ζ and $\tilde{\sigma}_0 = \sigma_0$, and $\mathcal{G}_\zeta^0(g) = 0$ otherwise. The following theorem is the main result of this section.

Theorem 5.2. *Let $\zeta \in \mathcal{Y}$, and for $\nu > 0$ let σ_ν satisfy conditions **(C1)**-**(C3q)** for any $q \in [2, +\infty)$ and let condition **(C4)** be satisfied. Then the solution $(u^\nu, \nu > 0)$ to (2.12) with initial condition ζ satisfies a large deviation principle in \mathcal{X} with the good rate function*

$$I(u) = \inf_{\{h \in L^2(0, T; H_0) : u = \mathcal{G}_\zeta^0(\int_0^\cdot h(s)ds)\}} \left\{ \frac{1}{2} \int_0^T |h(s)|_0^2 ds \right\}. \quad (5.4)$$

In order to prove this theorem, fix $q, p \in [4, \infty)$, $M > 0$ and $\nu_0 > 0$, let $(h_\nu, 0 < \nu \leq \nu_0)$ be a family of random elements taking values in the set \mathcal{A}_M defined by (2.13). Let $u_{h_\nu}^\nu$ be the solution of the following corresponding stochastic controlled equation

$$du_{h_\nu}^\nu(t) + [\nu Au_{h_\nu}^\nu(t) + B(u_{h_\nu}^\nu(t), u_{h_\nu}^\nu(t))]dt = \sqrt{\nu} \sigma_\nu(t, u_{h_\nu}^\nu(t)) dW(t) + \sigma_\nu(t, u_{h_\nu}^\nu(t)) h_\nu(t) dt, \quad (5.5)$$

with initial condition $u_{h_\nu}^\nu(0) = \xi \in \mathcal{Y}$. Note that $u_{h_\nu}^\nu = \mathcal{G}_\xi^\nu\left(\sqrt{\nu}(W + \frac{1}{\sqrt{\nu}} \int_0^\cdot h_\nu(s) ds)\right)$ due to the uniqueness of the solution. The following proposition establishes the weak convergence of the family $(u_{h_\nu}^\nu)$ as $\nu \rightarrow 0$.

Proposition 5.3. *Let us assume that for $\nu > 0$ the coefficients σ_ν satisfy conditions **(C1)**–**(C3q)** for all $q \in [2, +\infty)$ and that condition **(C4)** holds true. Let ζ be \mathcal{F}_0 -measurable such that $\mathbb{E}(|\zeta|_H^p + \|\zeta\|_{\mathcal{Y}}^p) < +\infty$ for every $p \in [2, \infty)$, and let h_ν converge to h in distribution as random elements taking values in \mathcal{A}_M , where this set is defined by (2.13) and endowed with the weak topology of the space $L_2(0, T; H_0)$. Then, as $\nu \rightarrow 0$, the solution $u_{h_\nu}^\nu$ of (5.5) converges in distribution in \mathcal{X} to the solution u_h^0 of (3.1). That is, as $\nu \rightarrow 0$, the process $\mathcal{G}_\zeta^\nu\left(\sqrt{\nu}(W + \frac{1}{\sqrt{\nu}} \int_0^\cdot h_\nu(s) ds)\right)$ converges in distribution to $\mathcal{G}_\zeta^0(\int_0^\cdot h(s) ds)$ in \mathcal{X} .*

Proof. Step 1: Let us decompose $u_{h_\nu}^\nu = \zeta + \sum_{i=1}^4 J_i$, where

$$\begin{aligned} J_1 &= -\nu \int_0^t Au_{h_\nu}^\nu(s) ds, & J_2 &= -\int_0^t B(u_{h_\nu}^\nu(s), u_{h_\nu}^\nu(s)) ds, \\ J_3 &= \sqrt{\nu} \int_0^t \sigma_\nu(s, u_{h_\nu}^\nu(s)) dW(s), & J_4 &= \int_0^t \sigma_\nu(s, u_{h_\nu}^\nu(s)) h_\nu(s) ds. \end{aligned}$$

For $\nu \in (0, \nu_0]$ we have using Minkowski's and Cauchy-Schwarz's inequalities

$$\begin{aligned} \|J_1\|_{W^{1,2}(0,T;H)}^2 &= \nu \int_0^T \left| \int_0^t Au_{h_\nu}^\nu(s) ds \right|_H^2 dt + \nu \int_0^T |Au_{h_\nu}^\nu(t)|_H^2 dt \\ &\leq C(T, p) \nu \int_0^T |Au_{h_\nu}^\nu(s)|_H^2 ds. \end{aligned}$$

Hence, using the estimate (2.23), we get that for $\nu \in (0, \nu_0]$,

$$\mathbb{E}\|J_1\|_{W^{1,2}(0,T;H)}^2 \leq \tilde{C}_1(M, T, \nu_0)[1 + \mathbb{E}\|\zeta\|^4]. \quad (5.6)$$

Similarly, the upper estimate (2.23) implies that for all $p \in [2, \infty)$ and $\nu \in (0, \nu_0]$,

$$\begin{aligned} \mathbb{E}\|J_1\|_{W^{1,p}(0,T;V')}^p &\leq \nu C(T) \mathbb{E} \int_0^T \|Au_{h_\nu}^\nu(s)\|_{V'}^p ds \\ &\leq \nu C(T) \mathbb{E} \int_0^T \|u_{h_\nu}^\nu(s)\|^p ds \leq C(T, p, \nu_0)[1 + \mathbb{E}\|\zeta\|^p]. \end{aligned} \quad (5.7)$$

Using again Minkowski's and Hölder's inequalities and the estimate (6.8), we deduce that for $4 \leq p < q < \infty$ and $\nu \in (0, \nu_0]$,

$$\|J_2\|_{W^{1,p}(0,T;H)}^p \leq C(T, p, \nu_0) \int_0^T \|u_{h_\nu}^\nu(t)\|_{H^{1,q}}^p \|u_{h_\nu}^\nu(t)\|^p dt.$$

Thus Hölder's inequality with the conjugate exponents q/p and $q/(q-p)$ and the upper estimates (2.23) and (4.1) yield for $\nu \in (0, \nu_0]$:

$$\mathbb{E}\|J_2\|_{W^{1,p}(0,T;H)}^p \leq C(T, M, p, q)[1 + \mathbb{E}\|\zeta\|^{pq/(q-p)}]^{1-p/q} [1 + \mathbb{E}\|\zeta\|_{H^{1,q}}^q]^{p/q}. \quad (5.8)$$

The Minkowski and Cauchy Schwarz inequalities and condition **(C1)** imply for $\nu \in (0, \nu_0]$

$$\begin{aligned} \|J_4\|_{W^{1,2}(0,T;H)}^2 &\leq C(T) \int_0^T \|\sigma_\nu(s, u_{h_\nu}^\nu(s))\|_{L_Q}^2 |h_\nu(s)|_0^2 ds \\ &\leq C(T, \nu_0, M) \left[1 + \sup_{0 \leq t \leq T} |u_{h_\nu}^\nu(t)|_H^2 \right]. \end{aligned}$$

Thus the upper estimate (2.18) yields that for $\nu \in (0, \nu_0]$ one has:

$$\mathbb{E} \|J_4\|_{W^{1,2}(0,T;H)}^2 \leq C(T, M) [1 + \mathbb{E} |\zeta|_H^4]. \quad (5.9)$$

Furthermore, Hölder's inequality and **(C1)** imply that for $\nu \in (0, \nu_0]$ and $p \in [4, \infty)$:

$$\int_0^T |J_4(t)|_H^p dt \leq M^{\frac{p}{2}} C [1 + \sup_{s \leq T} |u_{h_\nu}^\nu(s)|_H^p].$$

Let $\alpha \in (0, \frac{1}{2})$; then using again Minkowski's and Hölder's inequalities, condition **(C1)** and Fubini's theorem, we deduce that for $\nu \in (0, \nu_0]$:

$$\begin{aligned} &\int_0^T \int_0^T \frac{|J_4(t) - J_4(s)|_H^p}{(t-s)^{1+\alpha p}} ds dt \\ &\leq 2 \int_0^T dt \int_0^t ds (t-s)^{-1-\alpha p} \left| \int_s^t |\sigma_\nu(r, u_{h_\nu}^\nu(r))|_{L_Q} |h_\nu(r)|_0 dr \right|^p \\ &\leq C M^{\frac{p}{2}} [1 + \sup_{r \leq T} |u_{h_\nu}^\nu(r)|_H^p] \int_0^T dt \int_0^t (t-s)^{-1+(1/2-\alpha)p} ds. \end{aligned}$$

The two above estimates and (2.18) imply that for $\alpha \in (0, \frac{1}{2})$, $p \in [4, \infty)$ and $\nu \in (0, \nu_0]$:

$$\mathbb{E} \|J_4\|_{W^{\alpha,p}(0,T;H)}^p \leq C(p, \alpha, T, M) [1 + \mathbb{E} |\zeta|_H^p]. \quad (5.10)$$

The Burkholder-Davis-Gundy and Hölder inequalities imply

$$\begin{aligned} \mathbb{E} \int_0^T |J_3(t)|_H^p dt &\leq C_p \nu^{p/2} \int_0^T \mathbb{E} \left(\int_0^t |\sigma_\nu(s, u_{h_\nu}^\nu(s))|_{L_Q}^2 ds \right)^{p/2} dt \\ &\leq C_p T^{p/2-1} \nu^{p/2} \int_0^T \mathbb{E} |\sigma_\nu(s, u_{h_\nu}^\nu(t))|_{L_Q}^p dt. \end{aligned}$$

Let $p \in [4, \infty)$, $\alpha \in (0, \frac{1}{2})$ and for $t \in [0, T]$ set $\phi(t) := \int_0^t |\sigma_\nu(s, u_{h_\nu}^\nu(s))|_{L_Q}^2 ds$; then the Burkholder-Davis-Gundy and Hölder inequalities imply

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^T \frac{|J_3(t) - J_3(s)|_H^p}{|t-s|^{1+p\alpha}} dt ds &= \nu^{p/2} \int_0^T dt \int_0^T ds \frac{\mathbb{E} |\int_{s \wedge t}^{s \vee t} \sigma_\nu(r, u_{h_\nu}^\nu(r)) dW(r)|_H^p}{|t-s|^{1+p\alpha}} \\ &\leq C_p \nu^{p/2} \int_0^T \int_0^T \mathbb{E} \left| \int_{s \wedge t}^{s \vee t} |\sigma_\nu(r, u_{h_\nu}^\nu(r))|_{L_Q}^2 dr \right|^{p/2} |t-s|^{-(1+p\alpha)} dt ds \\ &\leq C_p \nu^{p/2} \mathbb{E} \int_0^T \int_0^T |\phi(t) - \phi(s)|^{p/2} |t-s|^{-(1+p\alpha)} dt ds \\ &\leq C_p \nu^{p/2} \mathbb{E} \|\phi\|_{W^{2\alpha,p/2}(0,T;\mathbb{R})}^{\frac{p}{2}} \\ &\leq C_p \nu^{p/2} \mathbb{E} \|\phi\|_{W^{1,p/2}(0,T;\mathbb{R})}^{\frac{p}{2}} \\ &\leq C_p C(T) \nu^{p/2} \mathbb{E} \int_0^T |\sigma_\nu(s, u_{h_\nu}^\nu(s))|_{L_Q}^p ds. \end{aligned}$$

Using the assumption **(C1)** and the two above upper estimates of J_3 , we deduce that

$$\mathbb{E}\|J_3\|_{W^{\alpha,p}(0,T;H)}^p \leq C(p,T)\nu^{p/2} \left[1 + \sup_{0 \leq t \leq T} \mathbb{E}|u_{h_\nu}^\nu(t)|_H^p\right].$$

Finally, the upper estimate (2.18) yields for $\nu \in (0, \nu_0]$ and $p \in [4, \infty)$:

$$\mathbb{E}\|J_3\|_{W^{\alpha,p}(0,T;H)}^p \leq C(p,T)\nu_0^{p/2} [1 + \mathbb{E}|\zeta|_H^p]. \quad (5.11)$$

Collecting all the estimates (5.6)-(5.11) we deduce that for $p \in [4, \infty)$, $\alpha \in (0, 1/2)$, there exists a positive constant $C(p, M, T)$ such that for any $\nu \in (0, \nu_0]$

$$\mathbb{E}\|u_{h_\nu}^\nu\|_{W^{\alpha,2}(0,T;H)}^2 + \mathbb{E}\|u_{h_\nu}^\nu\|_{W^{\alpha,p}(0,T;V')}^p \leq C(p, M, T). \quad (5.12)$$

Step 2: The upper estimates (2.23) and (5.12) show that the process $(u_{h_\nu}^\nu, \nu \in (0, \nu_0])$ is bounded in probability in

$$W^{\alpha,2}(0, T; H) \bigcap L^2(0, T; V) \bigcap W^{\alpha,p}(0, T; V').$$

Thanks to the compactness theorem given in [30], Chapter 1, Section 5. the space $W^{\alpha,2}(0, T; H) \bigcap L^2(0, T; V)$ is compactly embedded in $L^2(0, T; \mathcal{H})$. For $p\alpha > 1$, thanks to Theorem 2.2 given in [25] (see also [11] and the references therein), the space $W^{\alpha,p}(0, T; V')$ is compactly embedded in $\mathcal{C}([0, T]; D(A^{-\beta}))$ with $2\beta > 1$.

On the other hand, the family (h_ν) is included in \mathcal{A}_M . Set $F_\nu(t) = \int_0^t h_\nu(s)ds$; since H_0 is compactly embedded in H , we can again use the above compact embedding theorem and deduce that $W^{1,2}(0, T; H_0)$ is compactly embedded in $\mathcal{C}([0, T]; H)$. Furthermore, by assumption $h_\nu \rightarrow h$ in distribution in $L^2(0, T; H_0)$ endowed with the weak topology. This yields that $F_\nu \rightarrow F$ in distribution in the weak topology of $W^{1,2}(0, T; H_0)$, denoted by $W^{1,2}(0, T; H_0)_w$, where $F(t) := \int_0^t h(s)ds$.

Hence, by the Prokhorov theorem, the family of distributions $(\mathcal{L}(h_\nu, u_{h_\nu}^\nu, \nu \in (0, \nu_0])$ of the process $(F_\nu, u_{h_\nu}^\nu, \nu \in (0, \nu_0])$ is tight in

$$\mathcal{Z} := \left[W^{1,2}(0, T; H_0)_w \bigcap \mathcal{C}([0, T], H) \right] \times \left[L^2(0, T; \mathcal{H}) \bigcap \mathcal{C}([0, T]; D(A^{-\beta})) \right].$$

Let $(\nu_n, n \geq 0)$ be a sequence in $(0, \nu_0]$ such that $\nu_n \rightarrow 0$. Thus, we can extract a subsequence, still denoted by $(F_{\nu_n}, u_{h_{\nu_n}}^{\nu_n})$, that converges in distribution in \mathcal{Z} to a pair (\bar{F}, \bar{u}) as $n \rightarrow \infty$. Note that by assumption, $\bar{F} = F$.

Step 3: By the Skorohod-Jakubowski Theorem, [26] Theorem 2, recalled in the Appendix (see also [9]), there exists a stochastic basis $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1), \mathbb{P}^1)$ and on this basis, \mathcal{Z} -valued random variables $(F^1 = \int_0^\cdot h^1(s)ds, u^1)$ and for $n \geq 0$ $(F^{\nu_n,1} = \int_0^\cdot h^{\nu_n,1}(s)ds, u_{h^{\nu_n,1}}^{\nu_n,1})$, such that the pairs (F^1, u^1) and (\bar{F}, \bar{u}) have the same distribution, for $n \geq 0$ the pairs $(F^{\nu_n,1}, u_{h^{\nu_n,1}}^{\nu_n,1})$ and $(F_{\nu_n}, u_{h_{\nu_n}}^{\nu_n})$ have the same distribution on \mathcal{Z} , and as $n \rightarrow \infty$, $(F^{\nu_n,1}, u_{h^{\nu_n,1}}^{\nu_n,1}) \longrightarrow (F^1, u^1)$ in \mathcal{Z} \mathbb{P}^1 a.s. To ease notations in the sequel, we will skip the upper index 1 and the index n of the subsequence and still denote $F^{1,\nu}$ by F_ν , $h^{1,\nu}$ by h_ν , $u_{h^{1,\nu}}^{1,\nu}$ by $u_{h_\nu}^\nu$, F^1 by F , h^1 by h and u^1 by \bar{u} . Let again ζ denote the initial condition $u_{h^{1,\nu}}^{1,\nu}(0)$. Moreover, by (2.18), (2.23) and (4.1) we deduce the existence of constants C_i such that for $\nu \in (0, \nu_0]$, $\alpha \in (0, 1/2)$ and $q \in [2, \infty)$:

$$\mathbb{E}_1 \left(\sup_{0 \leq t \leq T} |u_{h_\nu}^\nu(t)|_H^2 \right) \leq C_1, \quad \mathbb{E}_1 \int_0^T \|u_{h_\nu}^\nu(t)\|^2 dt \leq C_2, \quad \mathbb{E}_1 \left(\sup_{0 \leq t \leq T} \|u_{h_\nu}^\nu(t)\|_{H^{1,q}(D)}^q \right) \leq C_3.$$

Therefore, we can extract a further subsequence which converges weakly to \bar{u} in $L^2(\Omega^1 \times (0, T); V) \cap L^q(\Omega^1 \times (0, T); H^{1,q})$ as $n \rightarrow \infty$. This implies that

$$\bar{u} \in L^2(0, T; V) \cap L^\infty(0, T; H \cap H^{1,q}) \quad \mathbb{P}^1 - \text{a.s.} \quad (5.13)$$

Step 4: (Identification of the limit.) We have to prove that the limit \bar{u} is solution of the equation

$$d\bar{u}(t) + B(\bar{u}(t), \bar{u}(t)) dt = \sigma_0(t, \bar{u}(t)) h(t) dt, \quad \bar{u}(0) = \zeta. \quad (5.14)$$

Let $\varphi \in D(A^\beta)$ with $2\beta > 1$; then

$$(u_{h_\nu}^\nu(t) - \zeta, \varphi) + \int_0^t \langle B(\bar{u}(s), \bar{u}(s)) - \sigma_0(s, \bar{u}(s))h(s), \varphi \rangle ds = \sum_{1 \leq i \leq 6} I_i, \quad (5.15)$$

where

$$\begin{aligned} I_1 &= -\nu \int_0^t (Au_{h_\nu}^\nu(s), \varphi) ds, \quad I_2 = \sqrt{\nu} \int_0^t (\sigma_\nu(s, u_{h_\nu}^\nu(s)) dW(s), \varphi), \\ I_3 &= - \int_0^t [\langle B(u_{h_\nu}^\nu(s) - \bar{u}(s), u_{h_\nu}^\nu(s)), \varphi \rangle + \langle B(\bar{u}(s), u_{h_\nu}^\nu(s) - \bar{u}(s)), \varphi \rangle] ds, \\ I_4 &= \int_0^t ([\sigma_\nu(s, u_{h_\nu}^\nu(s)) - \sigma_0(s, u_{h_\nu}^\nu(s))] h_\nu(s), \varphi) ds, \\ I_5 &= \int_0^t ([\sigma_0(s, u_{h_\nu}^\nu(s)) - \sigma_0(s, \bar{u}(s))] h_\nu(s), \varphi) ds, \\ I_6 &= \int_0^t (\sigma_0(s, \bar{u}(s)) [h_\nu(s) - h(s)], \varphi) ds. \end{aligned}$$

Since $\beta \geq 1/2$ implies that $\text{Dom}(A^\beta) \subset V$, using Cauchy-Schwarz's inequality and (2.23), we deduce for $t \in [0, T]$ and $\nu \in (0, \nu_0]$:

$$\begin{aligned} \mathbb{E}_1 |I_1| &\leq \nu \mathbb{E}_1 \int_0^t \|u_{h_\nu}^\nu(s)\| \|\varphi\| ds \leq \nu \sqrt{t} \|\varphi\| \left(\mathbb{E}_1 \int_0^t \|u_{h_\nu}^\nu(s)\|^2 ds \right)^{1/2} \\ &\leq \nu C(T, M) \|\varphi\| [1 + \mathbb{E} \|\zeta\|^4]^{1/2}. \end{aligned} \quad (5.16)$$

The Itô isometry, the Cauchy-Schwarz's inequality, condition **(C1)** and (2.18) yield

$$\begin{aligned} \mathbb{E}_1 |I_2| &\leq \sqrt{\nu} \mathbb{E}_1 \left(\int_0^t |\sigma_\nu(s, u_{h_\nu}^\nu(s))|_{L_Q}^2 \|\varphi\|^2 ds \right)^{1/2} \\ &\leq \sqrt{\nu} \|\varphi\| C(T, M) [1 + \mathbb{E} \|\zeta\|_H^4]^{1/2}. \end{aligned} \quad (5.17)$$

Using (2.8), the Cauchy-Schwarz inequality and (2.23) we get

$$\begin{aligned} \mathbb{E}_1 |I_3| &\leq C \mathbb{E} \int_0^t \|u_{h_\nu}^\nu(s) - \bar{u}(s)\|_{\mathcal{H}} (\|u_{h_\nu}^\nu(s)\|_{\mathcal{H}} + \|\bar{u}(s)\|_{\mathcal{H}}) \|\varphi\| ds \\ &\leq C \|\varphi\| \left(\mathbb{E}_1 \int_0^t \|u_{h_\nu}^\nu(s) - \bar{u}(s)\|_{\mathcal{H}}^2 ds \right)^{1/2} \left(\mathbb{E}_1 \int_0^t [\|u_{h_\nu}^\nu(s)\|_V^2 + \|\bar{u}(s)\|_V^2] ds \right)^{1/2} \\ &\leq C(T, M) \|\varphi\| [1 + \mathbb{E} \|\zeta\|^4]^{1/2} \left(\mathbb{E}_1 \int_0^t \|u_{h_\nu}^\nu(s) - \bar{u}(s)\|_{\mathcal{H}}^2 ds \right)^{1/2}. \end{aligned} \quad (5.18)$$

Using assumption **(C4)**, the Cauchy Schwarz inequality and (2.18) we obtain

$$\mathbb{E}_1 |I_4| \leq \mathbb{E}_1 \int_0^t |\sigma_\nu(s, u_{h_\nu}^\nu(s)) - \sigma_0(s, u_{h_\nu}^\nu(s))|_{L(H_0, H)} |h_\nu(s)|_0 \|\varphi\|_H ds$$

$$\begin{aligned}
&\leq |\varphi|_H \sqrt{MT} \left(\mathbb{E}_1 \int_0^t |\sigma_\nu(s, u_{h_\nu}^\nu(s)) - \sigma_0(s, u_{h_\nu}^\nu(s))|_{L(H_0, H)}^2 ds \right)^{1/2} \\
&\leq C(\nu) |\varphi|_H \sqrt{MT} \left(\mathbb{E}_1 \int_0^t [1 + |u_{h_\nu}^\nu(s)|_H^2] ds \right)^{1/2} \\
&\leq C(\nu) |\varphi|_H C(T, M) [1 + \mathbb{E}_1 |\zeta|_H^2]^{1/2}.
\end{aligned} \tag{5.19}$$

Condition **(C1)** and the Cauchy Schwarz inequality yield the existence of $\bar{L}_1 > 0$ such that for $\nu \in (0, \nu_0]$

$$\begin{aligned}
\mathbb{E}_1 |I_5| &\leq \mathbb{E}_1 \int_0^t |\sigma_0(s, u_{h_\nu}^\nu) - \sigma_0(s, \bar{u}(s))|_{L(H_0, H)} |h_\nu(s)|_0 |\varphi|_H ds \\
&\leq |\varphi|_H \sqrt{MT} \left(\mathbb{E}_1 \int_0^t |\sigma_0(s, u_{h_\nu}^\nu(s)) - \sigma_0(s, \bar{u}(s))|_{L_Q}^2 ds \right)^{1/2} \\
&\leq |\varphi|_H \sqrt{MT} \sqrt{\bar{L}_1} \left(\mathbb{E}_1 \int_0^t |u_{h_\nu}^\nu(s) - \bar{u}(s)|_H^2 ds \right)^{1/2}.
\end{aligned} \tag{5.20}$$

Finally, we have that

$$\mathbb{E}_1 |I_6| = \mathbb{E}_1 \left| \int_0^t ([h_\nu(s) - h(s)], \sigma_0^*(s, \bar{u}(s))\varphi) ds \right|. \tag{5.21}$$

Using the upper estimates (5.16), (5.17), (5.19) and (5.20) we deduce that $\mathbb{E}_1 |I_i| \rightarrow 0$ for $i = 1, 2, 4$ as $n \rightarrow \infty$ and $\nu_n \rightarrow 0$. Furthermore, by construction, we have \mathbb{P}^1 a.s. $u_{h_{\nu_n}}^{\nu_n} - \bar{u} \rightarrow 0$ in $L^2(0, T; H^{1,4})$ and hence in $L^2(0, T; \mathcal{H})$ and in $L^2(0, T; H)$ as $n \rightarrow \infty$. Furthermore, the estimates (2.23), (3.3) prove that $\int_0^T \|u_{h_\nu}^\nu(s) - \bar{u}(s)\|^2 ds$ is bounded in $L^2(\mathbb{P}^1)$ and hence is uniformly integrable. Therefore, the dominated convergence theorem and (5.18) prove that $\mathbb{E}_1 |I_i| \rightarrow 0$ for $i = 3, 5$. Finally, condition **(C1)** shows that

$$\int_0^T |\sigma_0^*(s, \bar{u}(s))\varphi|_0^2 ds \leq |\varphi|_H^2 \int_0^T [\bar{K}_0 + \bar{K}_1 |\bar{u}(s)|_H^2] ds$$

and by assumption, as $n \rightarrow \infty$, we have $h_{\nu_n} - h \rightarrow 0$ in $L^2(0, T; H_0)$ for the weak topology \mathbb{P}^1 a.s. Hence \mathbb{P}^1 a.s., $\int_0^t ([h_{\nu_n}(s) - h(s)], \sigma_0^*(s, \bar{u}(s))\varphi) ds$ converges to 0 as $n \rightarrow \infty$. Furthermore, the upper estimate (3.3) proves that this family is bounded in $L^2(\mathbb{P}^1)$; using once more the dominated convergence theorem, (5.21) proves that $\mathbb{E}_1 |I_6| \rightarrow 0$ as $n \rightarrow \infty$. Thus, (5.15) shows that as $n \rightarrow \infty$, for any $t \in [0, T]$ and $\varphi \in D(A^\beta)$ with $\beta > 1/2$:

$$\mathbb{E}_1 \left[(u_{h_{\nu_n}}^{\nu_n}(t), \varphi) - \int_0^t \langle -B(\bar{u}(s), \bar{u}(s)) + \sigma_0(s, \bar{u}(s))h(s), \varphi \rangle ds \right] \rightarrow 0. \tag{5.22}$$

On the other hand, by construction, since $\varphi \in \text{Dom}(A^\beta)$, we have \mathbb{P}^1 a.s.

$$\sup_{t \in [0, T]} |(u_{h_{\nu_n}}^{\nu_n}(t) - \bar{u}(t), \varphi)| \rightarrow 0 \quad \mathbb{P}^1 \text{ a.s. as } \nu \rightarrow 0.$$

Using again (2.23), (3.3) and the dominated convergence theorem, we deduce that as $n \rightarrow \infty$,

$$\mathbb{E}_1 \left(\sup_{t \in [0, T]} |(u_{h_{\nu_n}}^{\nu_n}(t) - \bar{u}(t), \varphi)| \right) \rightarrow 0. \tag{5.23}$$

Since \mathbb{P}^1 a.s. $\bar{u} \in \mathcal{C}([0, T], D(A^{-\beta}))$, this identity holds a.s. for all $t \in [0, T]$ and \bar{u} is a solution to the inviscid evolution equation (3.1). Thus the uniqueness of the solution to (3.1) proved in Theorem 3.2 implies that $\bar{u} = u_h^0$. Theorems 3.1 and 3.2 prove that u_h^0 belongs to $C([0, T]; H) \cap L^\infty(0, T; V \cap H^{1,q})$. Hence, from any sequence $\nu_n \rightarrow 0$, one can

extract a subsequence $(\nu_{n_k}, k \geq 0)$ such that $u_{h\nu_{n_k}}^{\nu_{n_k}} \rightarrow u_h^0$ in distribution in \mathcal{X} . This implies that the family $u_{h\nu}^\nu$ converges to u_h^0 in distribution in \mathcal{H} , which concludes the proof. \square

The following compactness result is the second ingredient which allows to transfer the LDP from $\sqrt{\nu}W$ to u^ν .

Proposition 5.4. *Suppose that $\tilde{\sigma}_0$ satisfies condition **(C1Bis)**, **(C2Bis)** and **(C3qBis)** for all $q \in [2, +\infty)$. Fix $M > 0$, $\zeta \in \mathcal{Y}$ and let $K_M = \{u_h^0 : h \in S_M\}$, where u_h^0 is the unique solution in \mathcal{X} of the deterministic control equation (3.1). Then K_M is a compact subset of \mathcal{X} .*

Proof. To simplify the notation, we skip the superscript 0 which refers to the inviscid case. By Theorems 3.1 and 3.2, $K_M \subset \mathcal{X}$. Let $(u_n, n \geq 1)$ be a sequence in K_M , corresponding to solutions of (3.1) with controls $(h_n, n \geq 1)$ in S_M :

$$du_n(t) + B(u_n(t), u_n(t))dt = \tilde{\sigma}_0(t, u_n(t))h_n(t)dt, \quad u_n(0) = \zeta.$$

Since S_M is a bounded closed subset of the Hilbert space $L^2(0, T; H_0)$, it is weakly compact. So there exists a subsequence of (h_n) , still denoted as (h_n) , which converges weakly to a limit $h \in L^2(0, T; H_0)$. Note that in fact $h \in S_M$ as S_M is closed.

We at first prove that (u_n) is bounded in $W^{1,2}(0, T; L^q) \cap W^{\alpha,p}(0, T; L^q) \cap L^2(0, T; H^{1,q})$ for any $p, q > 2$ and $\alpha < \frac{1}{2}$. Indeed, $u_n(t) = \zeta + J_1(t) + J_2(t)$, where

$$J_1(t) = - \int_0^t B(u_n(s), u_n(s))ds, \quad J_2(t) = \int_0^t \tilde{\sigma}_0(s, u_n(s))h_n(s)ds.$$

Hölder's inequality, (6.8), and (3.10) yield

$$\|J_1\|_{W^{1,q}(0,T,L^q)}^q \leq C(T) \sup_{t \in [0,T]} \|u_n(t)\|_{H^{1,q}}^{2q} \leq q^{2q}C(T, M)[1 + \|\zeta\| + \|\operatorname{curl} \zeta\|_q]^{2q}. \quad (5.24)$$

Furthermore, Minkowski's inequality, the Sobolev embedding theorem (see (6.1)), (6.3), condition **(C2Bis)** and (3.3) yield

$$\begin{aligned} \|J_2\|_{W^{1,2}(0,T,L^q)}^2 &\leq C(T, q) \int_0^T \|\tilde{\sigma}_0(t, u_n(t))h_n(t)\|_q^2 dt \leq C(T, q) \int_0^T \|\tilde{\sigma}_0(t, u_n(t))h_n(t)\|^2 dt \\ &\leq C(T, q) \int_0^T |\operatorname{curl} \tilde{\sigma}_0(t, u_n(t))|_{L(H_0, H)}^2 |h_n(t)|_0^2 dt \leq C(T, q) M [\tilde{K}_0 + \tilde{K}_1 \sup_{t \in [0,T]} \|u(t)\|^2] \\ &\leq C(T, q, M)[1 + \|\zeta\|^2]. \end{aligned} \quad (5.25)$$

The Minkowski and Hölder inequalities, the Sobolev embedding theorem, (6.3), conditions **(C1Bis)**, **(C2Bis)**, (3.3) and (6.1) imply

$$\begin{aligned} \int_0^T \|J_2(t)\|_q^p dt &\leq \int_0^T \left| \int_0^t \|\tilde{\sigma}_0(s, u_n(s))h_n(s)\|_q ds \right|^p dt \leq C(q) \int_0^T \left| \int_0^t \|\tilde{\sigma}_0(s, u_n(s))h_n(s)\| ds \right|^p dt \\ &\leq C(q)(MT)^{p/2} \sup_{t \in [0,T]} \|\tilde{\sigma}_0(t, u_n(t))\|_{L(H_0, V)}^p \leq C(p, q, T, M)[1 + \sup_{t \in [0,T]} \|u(t)\|^p] \\ &\leq C(p, q, T, M)[1 + \|\zeta\|^p]. \end{aligned} \quad (5.26)$$

Finally, similar arguments imply that for $\alpha \in (0, \frac{1}{2})$, we have

$$\begin{aligned} &\int_0^T \int_0^T \frac{\|J_2(t) - J_2(s)\|_q^p}{(t-s)^{1+\alpha p}} ds dt \\ &\leq 2C(q) \int_0^T dt \int_0^t ds (t-s)^{-1-\alpha p} \left| \int_s^t \|\tilde{\sigma}_0(r, u_n(r))h_n(r)\| ds \right|^p \end{aligned}$$

$$\begin{aligned}
&\leq 2C(q)(TM)^{\frac{p}{2}} C \left[1 + \sup_{r \in [0, T]} \|u_n(r)\|^p\right] \int_0^T dt \int_0^t (t-s)^{-1+(1/2-\alpha)p} ds \\
&\leq C(q, T, M) [1 + \|\zeta\|^p].
\end{aligned} \tag{5.27}$$

As in the proof of Proposition 5.3, Step 3, using [30] we deduce from the upper estimates (5.24)-(5.27) that the sequence (u_n) is relatively compact in $L^2(0, T; \mathcal{H}) \cap \mathcal{C}([0, T], D(A^{-\beta}))$ with $2\beta > 1$. Hence there exists a subsequence, still denoted (u_n) , which converges in $L^2(0, T; \mathcal{H}) \cap \mathcal{C}([0, T], D(A^{-\beta}))$ to some element u . It remains to check that u is the solution to the evolution equation

$$du(t) + B(u(t), u(t))dt = \tilde{\sigma}_0(t, u(t))h(t)dt, \quad u(0) = \zeta.$$

The proof, which is similar to that of Step 4 in Proposition 5.3 and easier, is briefly sketched. Only (deterministic) terms similar to I_i for $i = 3, 5$ and 6 have to be dealt with. As in the proof of Proposition 5.3, these terms are estimated replacing the upper estimate (2.23) by (3.3). This concludes the proof of the Proposition. \square

The proof of Theorem 5.2 is a straightforward consequence of Propositions 5.3 and 5.4, as shown in [13].

6. APPENDIX

6.1. Properties of the bilinear operator. Let us at first recall the following classical Sobolev embeddings which hold since D is a bounded domain of \mathbb{R}^2 which satisfies the cone condition (see e.g. [1]):

$$\|u\|_q \leq C(q)\|u\|_{W^{1,2}} \quad \text{for } u \in W^{1,2} \quad \text{and } 1 \leq q < +\infty, \tag{6.1}$$

$$W^{2,1} \subset \mathcal{C}_B^0(D), \quad W^{1,q} \subset \mathcal{C}_B^0(D) \quad \text{for } q \in (2, \infty). \tag{6.2}$$

Furthermore, recall the following result proved in [27] (see also [11] and [40] for the way the constant depends on q). Given $q \in [2, \infty)$ there exists a constant C such that for every $u \in H^{1,q}$ one has:

$$\|\nabla u\|_q \leq Cq \|\text{curl } u\|_q \quad \text{for } q \in [2, \infty). \tag{6.3}$$

Furthermore, given $q \in [2, \infty)$ and $r > 0$, the operator B has a unique extension to a continuous bilinear operator from $H^{1,q} \times H^{1,q}$ to $H^{-r,q}$ and the following estimates are satisfied for some constant C and all $u, v \in H^{1,q}$ resp. $\varphi, \psi \in D(A)$:

$$\|B(u, v)\|_{H^{-r,q}} \leq C \|u\|_{H^{1,q}} \|v\|_{H^{1,q}}, \tag{6.4}$$

$$\langle B(u, v), v \rangle = 0, \tag{6.5}$$

$$\langle \text{curl } B(\varphi, \varphi), \psi \rangle = \langle \varphi \cdot \nabla(\text{curl } \varphi), \psi \rangle = \langle B(\varphi, \text{curl } \varphi), \psi \rangle, \tag{6.6}$$

$$\langle \text{curl } B(u, v), \text{curl } v | \text{curl } v |^{q-2} \rangle = 0 \quad \text{for all } u, v \in H^{2,q} \cap D(A). \tag{6.7}$$

Finally, if $q > 2$, there exists a constant $C > 0$ such that for all $u, v \in H^{1,q}$

$$|B(u, v)|_H \leq C \|u\|_{H^{1,q}} \|v\|_{H^{1,2}} \quad \text{and} \quad \|B(u, v)\|_q \leq C \|u\|_{H^{1,q}} \|v\|_{H^{1,q}}. \tag{6.8}$$

6.2. Radonifying operators and stochastic calculus in $W^{k,q}$ spaces. In this section, we recall the basic definitions and results of stochastic calculus on non Hilbert Sobolev spaces used in this paper. Their proofs can be found in references [10], [11], [21], [36] and [37].

Let \mathcal{E} be a Banach space, such as the Sobolev spaces $W^{k,q}$ for $k \geq 0$ and $q \in [1, \infty)$, and let H_0 be a Hilbert space. The following notion extends that of Hilbert Schmidt operator from H_0 to \mathcal{E} when \mathcal{E} is not a Hilbert space. Let (e_k) denote an orthonormal basis of

H_0 and (β_k) be a sequence of independent standard Gaussian random variables on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Definition 6.1. A linear operator $K : H_0 \rightarrow \mathcal{E}$ is Radonifying if the series $\sum_k \beta_k K e_k$ converges in $L^2(\tilde{\Omega}, \mathcal{E})$. Let $R(H_0, \mathcal{E})$ denote the set of Radonifying operators, and given $K \in R(H_0, \mathcal{E})$, set

$$\|K\|_{R(H_0, \mathcal{E})} = \left(\tilde{\mathbb{E}} \left| \sum_k \beta_k K e_k \right|_{\mathcal{E}}^2 \right)^{\frac{1}{2}}. \quad (6.9)$$

Then $(R(H_0, \mathcal{E}), \|K\|_{R(H_0, \mathcal{E})})$ is a separable Banach space and $\|K\|_{R(H_0, \mathcal{E})}$ does not depend on the choice of (e_k) and (β_k) .

We now suppose that H_0 is the RKHS of the H -valued Wiener process $(W(t), t \geq 0)$ and fix some orthonormal basis (e_k) of H_0 . Simple $R(H_0, \mathcal{E})$ -valued processes σ on $[0, T]$ are defined as follows. Given integers $m, n \geq 1$, $0 \leq t_1 < t_2 < \dots < t_{m+1} \leq T$, and $(\sigma_j \in L^2(\Omega, \mathcal{F}_{t_j}; R(H_0, \mathcal{E})), j = 0, \dots, m)$ set

$$\sigma(t, \omega) := \sum_{0 \leq j \leq m} \sigma_j(\omega) 1_{(t_j, t_{j+1}]}(t).$$

For such a simple process σ , and $t \in (0, T)$, set

$$\int_0^t \sigma(s) dW_s := \sum_{0 \leq j \leq m} \sigma_j(\omega) Q^{\frac{1}{2}}(W(t_{j+1} \wedge t) - W(t_j \wedge t)).$$

The extension of stochastic integrals to predictable square integrable processes cannot be done for any Banach space \mathcal{E} . Fix $k \in [0, \infty)$ and $q \in [2, \infty)$ and let $\mathcal{E} = W^{k, q}$ (with the convention $L^q = W^{0, q}$). The stochastic integral can be extended uniquely as a linear bounded operator from the set of predictable processes in $L^2(0, T; R(H_0, H^{k, q}))$ to the set of (\mathcal{F}_t) adapted random variables in $L^2(\Omega, H^{k, q})$. Moreover, the following Burkholder-Davies-Gundy inequality holds (see e.g. [37], section 5): For any $p \in [1, \infty)$, there exists a constant $C_p > 0$ such that for any predictable process $\sigma \in L^2(0, T; R(H_0, H^{k, q}))$,

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s) dW_s \right|_{H^{k, q}}^p \right) \leq C_p \mathbb{E} \left(\int_0^T \|\sigma(s)\|_{R(H_0, H^{k, q})}^2 ds \right)^{\frac{p}{2}} \quad (6.10)$$

Finally, given $2 \leq q \leq p < \infty$, some predictable processes $\sigma \in L^2(0, T; R(H_0, H^{0, q}))$ and $f \in L^1(0, T; H^{0, q})$, we state a particular case of the Itô formula applied to the function $\Psi_{q, p}(\cdot) = \|\cdot\|_q^p$ on $H^{0, q}$ and the $H^{0, q}$ -valued process $(Z_t, t \in [0, T])$ defined by

$$Z(t) = Z(0) + \int_0^t \sigma(s) dW(s) + \int_0^t f(s) ds.$$

With the above notations, if $\langle F, G \rangle$ denotes the duality between $F \in L^q$ and $G \in L^{q*}$ with $q* = \frac{q}{q-1}$, we have:

$$\begin{aligned} \|Z(t)\|_q^p &= \|Z(0)\|_q^p + p \int_0^t \|Z(s)\|_q^{p-q} \langle |Z(s)|^{q-2} Z(s), f(s) \rangle ds \\ &\quad + p \int_0^t \|Z(s)\|_q^{p-q} \langle |Z(s)|^{q-2} Z(s), \sigma(s) dW(s) \rangle + \frac{1}{2} \int_0^t \text{tr}_{\sigma(s)} \Psi''_{q, p}(Z(s)) ds, \end{aligned} \quad (6.11)$$

and for every $u \in H^{0, q}$,

$$0 \leq \text{tr}_{\sigma(s)} \Psi''_{q, p}(u) \leq p(p-1) \|u\|_q^{p-2} \|\sigma(s)\|_{R(H_0, H^{0, q})}^2. \quad (6.12)$$

6.3. Nemytski operators. In this section we will show that assumptions **(C1)** – **(C3qBis)** are satisfied by Nemytski operators.

Definition 6.2. Let $q \in [2, \infty)$. A mapping $g : [0, T] \times D \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ belongs to the class $U(D, q)$ if and only if $g(t, x, y) = g^1(t, x) + g^2(t, x, y)$, $t \in [0, T]$, $x \in D$, $y \in \mathbb{R}^2$, where:

- (1) g^1 and g^2 are measurable, and for any $t \in [0, T]$, $g^1(t, \cdot) \in H^{1,2} \cap H^{1,q}$ and $g^2(t, \cdot, \cdot)$ is differentiable,
- (2) there are a constant $c > 0$ and $\phi \in L^2(D) \cap L^q(D)$ such that all $t \in [0, T]$, and $x \in D$, $y \in \mathbb{R}^2$,

$$|g^1(t, \cdot)|_{H^{1,2}} + |g^1(t, \cdot)|_{H^{1,q}} \leq c,$$

$$|g^2(t, x, y)| + \sum_{i=1,2} |\partial_{x_i} g^2(t, x, y)| \leq c(\phi(x) + |y|), \quad \sum_{i=1,2} |\partial_{y_i} g^2(t, x, y)| \leq c.$$

We say that $g : [0, T] \times D \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ belongs to the class $U(D, \infty)$ if and only if it is differentiable with respect to the second and third variables, and there is a constant $c > 0$ such that for all $t \in [0, T]$, $x \in D$, $y \in \mathbb{R}^2$:

$$|g(t, x, y)| + \sum_{i=1,2} |\partial_{x_i} g(t, x, y)| + \sum_{i=1,2} |\partial_{y_i} g(t, x, y)| \leq c.$$

Let g_i , $i = 1, \dots, m$ and \tilde{g} be in $U(D, q)$ and define the Nemytski operators

$$\tilde{\sigma}(t, u)(x) = \tilde{g}(t, x, u(x)), \quad \text{and} \quad \sigma(t, u)\psi(x) = \sum_{1 \leq i \leq m} g_i(t, x, u(x))\psi_i(x), \quad (6.13)$$

where $\psi_i \in H_0$, $i = 1, \dots, m$. These operators satisfy the assumptions **(C3q)** and **(C3qBis)** (see e.g. [11]). The condition $U(D, \infty)$ obviously implies $U(D, q)$ for every $q \in [2, \infty)$. Therefore, if the coefficients \tilde{g} and g_i belong to the class $U(D, \infty)$, then σ and $\tilde{\sigma}$ satisfy the conditions **(C1)**, **(C1Bis)**, **(C2)**, **(C2Bis)**, **(C3q)** and **(C3qBis)** for all $q \in [2, \infty)$.

6.4. The Skorohod-Jakubowski representation theorem. Let \mathcal{Z} be a topological space such that there exists a sequence (f_j) of continuous functions $f_j : \mathcal{Z} \rightarrow [-1, 1]$ that separate points of \mathcal{Z} .

The following result is proved in [26], Theorem 2.

Theorem 6.3. Let $(\mathcal{P}_j, j \in \mathbb{N})$ be a tight sequence of Borel probability measures on \mathcal{Z} . Then there exist a subsequence (j_k) and Borel measurable maps $\theta_k : [0, 1] \rightarrow \mathcal{Z}$, $k \geq 1$ such that for each $k \geq 1$, \mathcal{P}_{j_k} is equal to the law of θ_k and for every $s \in [0, 1]$, $\theta_k(s) \rightarrow \theta(s)$ in \mathcal{Z} as $k \rightarrow \infty$.

Acknowledgments: The work of H. Bessaih was partially supported by the NSF grant No. DMS 0608494. This work was partially written while the authors were staying at the Isaac Newton Institute for Mathematical Sciences in Cambridge. They would like to thank the Institute for the financial support, the very good working conditions and the friendly atmosphere.

REFERENCES

- [1] Adams, R. A. & J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics Series, 2nd edition, Academic Press, (2003)
- [2] Bardos, C. Existence et unicité de la solution de l'équation d'Euler en dimensions deux, *Jour. Math. Anal. Appl.* **40**, 769-780, (1972)
- [3] Bardos, C & Titi, E. S. Euler equations for an ideal incompressible fluid, (Russian) *Uspekhi Mat. Nauk* 62 no. 3(375), 5-46; translation in *Russian Math. Surveys* 62 no. 3, 409-451 (2007)

- [4] Bessaih, H. & Flandoli, F. 2-D Euler equations with non regular force, *Non Diff. Eq. Appl.*, **6**, Issue 1, 35-54 (1999)
- [5] Bessaih, H., Martingale solutions for stochastic Euler equations, *Stoc. Anal. Appl.*, **17** number 5, 713-727 (1999)
- [6] Bessaih, H. & Millet, A., Large deviation principle and inviscid shell models, *Electronic Journal of Probability*, **14-89**, 2051-2579 (2009)
- [7] Brzeźniak, Z. Stochastic partial differential equations in M -type 2 Banach spaces, *Potential Anal.*, **4**, 1-45 (1995)
- [8] Brzeźniak, Z., Capiński, M. & Flandoli, F., Stochastic Navier-Stokes equations with multiplicative noise, *Stoc. Anal. Appl.*, **10-5**, 523-532 (1992)
- [9] Brzeźniak, Z. & Ondreját, M, Weak Solutions to Stochastic Wave Equations with Values in Riemannian Manifolds, *Communications in Partial Differential Equations*, **36**, 16241653 (2011).
- [10] Brzeźniak, Z. & Peszat, S., Space-time continuous solutions to SPDEs driven by homogeneous Wiener spaces, *Studia Math.*, **137**, 261-299 (1999)
- [11] Brzeźniak, Z. & Peszat, S., Stochastic two dimensional Euler equations, *Ann. Probab.*, **29-4**, (2001), 1796-1832 (2001)
- [12] Budhiraja, A. & Dupuis, P., A variational representation for positive functionals of infinite dimensional Brownian motion, *Prob. and Math. Stat.* **20**, 39-61 (2000)
- [13] Budhiraja, A., Dupuis, P. & Maroulas, V., Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Prob.* **36**, 1390-1420 (2008)
- [14] Capiński, M. & Cutland, N.J., Stochastic Euler equations on the torus, *Ann. Appl. Probab.*, **9-3**, 688-705 (1999)
- [15] Capinsky, M. & Gatarek, D., Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension, *J. Funct. Anal.* **126** 26-35 (1994)
- [16] Cerrai, S. & Röckner, M., Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction terms, *Ann. Probab.* **32**, 1100-1139 (2004)
- [17] Chang, M.H., Large deviations for the Navier-Stokes equations with small stochastic perturbations. *Appl. Math. Comput.* **76**, 65-93 (1996)
- [18] Chueshov, I. & Millet, A., Stochastic 2D hydrodynamical type systems: Well posedness and large deviations, *Appl. Math. Optim.*, **61-3**, 379-420 (2010)
- [19] Da Prato, G. & Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, (1992)
- [20] Dembo, A. & Zeitouni, O, *Large Deviations Techniques and Applications*, Springer-Verlag, New York, (2000)
- [21] Dettweiler, E., Stochastic integration relative to Brownian motion on a general Banach space, *Doğa Math.*, **15**, 6-44 (1991)
- [22] A. Du, A., Duan, J. & Gao, H., Small probability events for two-layer geophysical flows under uncertainty, preprint arXiv:0810.2818, October 2008.
- [23] Duan, J. & Millet, A., Large deviations for the Boussinesq equations under random influences, *Stochastic Processes and their Applications* **119-6**, 2052-2081 (2009)
- [24] Dupuis, P., & Ellis, R.S., *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley-Interscience, New York, (1997)
- [25] F. Flandoli, F. & Gatarek, D., Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory relat. Fields*, **102**, 367-391 (1995).
- [26] Jakubowski, A, The almost sure Skorokhod representation for subsequences in nonmetric spaces, *Teor. Veroyatnost. i Primenen*, **4**, 209216; English translation in *Theory Probab. Appl.*, **42**, 167174, (1997).
- [27] Kato, T. & Ponce, G., Well-posedness of the Euler and Navier-Stokes equations in Lebesgue spaces $L^p_s(\mathbb{R}^n)$, *Rev. Mat. Iberoamericana* **2**, 73-88 (1986)
- [28] Kuksin, S. B., On the distribution of Energy and Vorticity for solutions of 2D Navier-Stokes Equation with Small Viscosity, *Commun. Math. Phys.*, **284**, 407-424 (2008).
- [29] Lions, J.L., *Équations Différentielles Opérationnelles et Problèmes aux Limites*, Springer-Verlag, Berlin (1961).
- [30] Lions, J.L., *Quelques Méthodes de Résolution de Problèmes aux Limites Non Linéaires*, Dunod, Paris (1970)
- [31] Liu, W., Large deviations for stochastic evolution equations with small multiplicative noise, *Appl. Math. Optim.*, **61**, no 1, 27-56 (2010).
- [32] Mariani, M., Large deviations principles for stochastic scalar conservation laws, *Probab. Theory and Related Fields*, **147**, no 3-4, 607-648 (2010)

- [33] Marcus, M. & Mizel, V. J., Nemitsky operators on Sobolev spaces, *Arch. Rational Mech. Anal.*, **51**, 347–370 (1973)
- [34] Menaldi, J.L., & Sritharan, S.S., Stochastic 2-D Navier-Stokes equation, *Appl. Math. Optim.* **46**, 31–53 (2002).
- [35] Mikulevicius, A., & Rozovskii, B., Stochastic Navier-Stokes equations for turbulent flows, *SIAM J. Math. Anal.*, **35** no 5, 1250–1310 (2004).
- [36] Neidhardt, A. L., Stochastic integrals in 2-uniformly smooth Banach spaces, Ph. D. dissertation, University of Wisconsin
- [37] Ondrejat, M., Uniqueness for stochastic evolution equations in Banach spaces, *Dissertationes Math. (Roaprawny Math.)*, **426**, 63 pages (2004)
- [38] Röckner, M. & Zhang, T., Large deviations for stochastic tamed 3D Navier-Stokes equations, *Appl. Math. Optim.*, **61**, no. 2, 267–285 (2010).
- [39] Sritharan, S.S., & P. Sundar, P., Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise, *Stoch. Proc. and Appl.* **116**, 1636–1659 (2006).
- [40] Yudovich, V.L., Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid, *Mathematical Research Letters*, **2**, 27–38 (1995)
- [41] Ziane, M., On the two-dimensional Navier-Stokes equations with the free boundary conditions, *App. Math. Optim.*, **38**, no 1, 1–19 (1998)

UNIVERSITY OF WYOMING, DEPARTMENT OF MATHEMATICS, DEPT. 3036, 1000 EAST UNIVERSITY AVENUE, LARAMIE WY 82071, UNITED STATES

E-mail address: `bessaih@uwyo.edu`

SAMM, EA 4543, UNIVERSITÉ PARIS 1 PANTHÉON SORBONNE, 90 RUE DE TOLBIAC, 75634 PARIS CEDEX FRANCE and LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, UNIVERSITÉS PARIS 6-PARIS 7, BOÎTE COURRIER 188, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE

E-mail address: `annie.millet@univ-paris1.fr` and `annie.millet@upmc.fr`